ABOUT THE CONDITIONS OF REPRESENTABILITY OF THE POLYNOMIAL NINTH DEGREE IN THE FORM OF SUPERPOSITION OF THE CUBIC POLYNOMIAL FROM THE CUBIC POLYNOMIAL

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The work is relevant in that it is possible to obtain a convenient method for solving an algebraic equation of the ninth degree and to answer the question whether this equation is solvable in radicals.

The purpose of the article is to formulate and justify the necessary and sufficient conditions for the representability of a ninth-degree algebraic polynomial in the form of a superposition of a cubic polynomial from a cubic polynomial.

Material and methods. The research material is an algebraic polynomial of the ninth degree of the form:

$$P_9(z) = z^9 + a_1 z^8 + a_2 z^7 + a_3 z^6 + a_4 z^5 + a_5 z^4 + a_6 z^3 + a_7 z^2 + a_8 z + a_9,$$

as well as representing this polynomial as a superposition of a cubic polynomial from a cubic polynomial.

And as the research methods, we used the methods of algebra, mathematical analysis and the Maple 2017 computer mathematics system.

Findings and their discussion. Let the algebraic equation be presented in the following form:

$$P_9(z) = z^9 + a_1 z^8 + a_2 z^7 + a_3 z^6 + a_4 z^5 + a_5 z^4 + a_6 z^3 + a_7 z^2 + a_8 z + a_9, \tag{1}$$

where $P_9(z)$ is a polynomial of the ninth degree, representable as a superposition of a cubic polynomial from a cubic polynomial. That is, it takes the following form:

$$P_9(z) = f_2\left[f_1(z)\right], \tag{2}$$

where $f_1(z) = z^3 + b_1 z^2 + b_2 z + b_3, \quad f_2(z) = z^3 + c_1 z^2 + c_2 z + c_3$.

Consider the problem for a polynomial of the ninth degree. As

$$\begin{aligned} f_2\left[f_1(z)\right] = {} & z^9 + 3b_1 z^8 + \left(3b_1^2 + 3b_2\right) z^7 + \left(b_1^3 + 6b_1 b_2 + 4b_3 + c_1\right) z^6 + \left(b_2^2 + \right. \\ & \left. + b_1\left(2b_1 b_2 + 2b_3\right) + b_2\left(b_1^2 + 2b_2\right) + 4b_1 b_3 + 2b_1 c_1\right) z^5 + \left(2b_2 b_3 + b_1\left(2b_1 b_3 + b_2^2\right) + \right. \\ & + b_2\left(2b_2 b_1 + 2b_3\right) + b_3\left(b_1^2 + 2b_2\right) + c_1\left(b_1^2 + 2b_2\right) z^4 + \left(b_2\left(2b_3 b_1 + b_2^2\right) + c_2 + \right. \\ & \left. + b_3\left(2b_2 b_1 + 2b_3\right) + c_1\left(2b_2 b_1 + 2b_3\right) + 2b_1 b_2 b_3 + b_3^2\right) z^3 + \left(b_3\left(2b_1 b_3 + b_2^2\right) + b_1 c_2 + \right. \\ & \left. + \left(c_1 + b_3\right)\left(2b_1 b_3 + b_2^2\right) + 2b_2^2 b_3 + b_1 b_3^2\right) z^2 + \left(3b_3^2 b_2 + 2c_1 b_2 b_3 + c_2 b_2\right) z + \\ & + b_3^3 + c_1 b_3^2 + c_2 b_3 + c_3, \end{aligned}$$

then equating the coefficients of the resulting polynomial and algebraic polynomial (1) we obtain a system of equations. After analyzing it, we obtain the following equalities:

$$a_4 = -\frac{5}{9}a_1^2a_2 + \frac{2}{3}a_1a_3 + \frac{10}{81}a_1^4 + \frac{1}{3}a_2^2; \tag{3}$$

$$a_5 = -\frac{1}{9}a_1^2a_3 - \frac{2}{243}a_1^5 + \frac{10}{81}a_1^3a_2 - \frac{1}{3}a_1a_2^2 + \frac{2}{3}a_2a_3; \tag{4}$$

$$\begin{aligned} a_7 = {} & \frac{16}{2187}a_1^7 + \frac{1}{27}a_1^4a_3 - \frac{41}{729}a_1^5a_2 + \frac{32}{243}a_1^3a_2^2 + \\ & + \frac{1}{9}a_2^2a_3 - \frac{7}{81}a_1a_2^3 - \frac{4}{27}a_1^2a_2a_3 + \frac{1}{3}a_1a_6; \end{aligned} \tag{5}$$

$$\begin{aligned} a_8 = {} & -\frac{1}{6561}\left(a_1^2 - 3a_2\right)\left(11a_1^6 - 75a_1^4a_2 + 54a_1^3a_3 + \right. \\ & \left. + 135a_1^2a_2^2 - 162a_1a_2a_3 - 27a_2^3 + 729a_6\right); \end{aligned} \tag{6}$$

the coefficient b remains arbitrary, and the rest have the following form:

$$b_1 = \frac{1}{3}a_1; \tag{7}$$

$$b_2 = \frac{1}{3}a_2 - \frac{1}{9}a_1^2; \tag{8}$$

$$b_3 = \frac{1}{3}a_3 + \frac{5}{81}a_1^3 - \frac{2}{9}a_1a_2 - \frac{1}{3}c_1; \tag{9}$$

$$\begin{aligned} c_2 = {} & a_6 - \frac{4}{81}a_1^3a_3 - \frac{1}{3}a_3^2 + \frac{1}{3}c_1^2 + \frac{2}{9}a_1a_2a_3 - \\ & - \frac{5}{243}a_1^4a_2 - \frac{1}{27}a_2^3 + \frac{8}{2187}a_1^6 + \frac{1}{27}a_1^2a_2^2 \end{aligned} \tag{10}$$

$$\begin{aligned} c_3 = {} & \frac{c_1^3}{27} + \frac{1}{81}\left(27a_6 - a_2^3 - 9a_3^2\right)c_1 + \frac{2}{27}a_3^3 + \frac{1}{81}\left(a_2^3 - 27a_6\right)a_3 - \\ & - \frac{2}{243}a_2\left(a_2^3 + 9a_3^2 - 9a_3c_1 - 27a_6\right)a_1 - \frac{1}{81}a_2^2\left(a_3 - c_1\right)a_1^2 + \\ & + \left(-\frac{5}{81}a_6 + \frac{4}{243}a_3^2 + \frac{47}{2187}a_2^3 - \frac{4}{243}a_3c_1\right)a_1^3 + \frac{1}{729}a_2\left(7a_3 - 5c_1\right)a_1^4 - \\ & - \frac{35}{2187}a_2^2a_1^5 + \frac{1}{6561}\left(8c_1 - 13a_3\right)a_1^6 + \frac{91}{19683}a_2a_1^7 - \frac{245}{531441}a_1^9 \end{aligned} \tag{11}$$

When calculating the expression $f_2\left[f_1(z)\right]$ with such coefficients, we obtain the identity

$$f_2\left[f_1(z)\right] = z^9 + a_1z^8 + a_2z^7 + a_3z^6 + a_4z^5 + a_5z^4 + a_6z^3 + a_7z^2 + a_8z + a_9, \qquad (12)$$

in which the coefficients $a_4,\ a_5,\ a_7,\ a_8$ are expressed by equalities (3) – (6). Thus, the following theorem is proved.

Theorem. Necessary and sufficient conditions for the representation of $f_2\left[f_1(z)\right]$ polynomial (1) in the form are equalities (3) – (6).

Consequence. Equalities (3) – (6) are sufficient conditions for the solvability of an equation $P_9(z)=0$ in radicals.

Conclusion. As a result of the study, the necessary and sufficient conditions for representing the polynomial of the ninth degree, in the form of a superposition of a cubic polynomial from a cubic polynomial, are found.