МАТЭМАТЫКА

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ПОСТРОЕНИЕ ЛИНЕЙНЫХ ФУНКЦИОНАЛЬНЫХ НАБЛЮДАТЕЛЕЙ СОСТОЯНИЯ ДЛЯ ЛИНЕЙНЫХ ПОЛОЖИТЕЛЬНЫХ ДИНАМИЧЕСКИХ СИСТЕМ ДРОБНОГО ПОРЯДКА С ЗАПАЗДЫВАНИЕМ И НЕИЗВЕСТНЫМИ ВХОДАМИ

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Статья посвящена задаче построения положительных функциональных наблюдателей для линейных положительных динамических систем дробного порядка с запаздыванием, входы которых неизвестны. Предложенные функциональные наблюдатели являются положительными, т.е. они гарантируют положительные оценки в произвольный момент времени. Кроме того, в терминах линейного программирования сформулированы необходимые и достаточные условия существования таких положительных функциональных наблюдателей.

Цель работы – построение положительных функциональных наблюдателей для линейных динамических систем дробного порядка с запаздыванием и неизвестными входами.

Материал и методы. *Материалом исследования являются функциональные наблюдатели для линейных систем дробного порядка с запаздыванием. Использованы методы математического анализа и линейной алгебры, а также численного моделирования.*

Результаты и их обсуждение. *Доказаны две теоремы. В теореме 1 утверждается, что исследуемый функциональный наблюдатель является положительным и линейным при определенных предположениях. В теореме 2 доказывается, что построенный наблюдатель восстанавливает матрицы наблюдения, если соответствующая задача линейного программирования разрешима. Также представлены три численных примера, демонстрирующих эффективность полученных результатов.*

Заключение. *Предложены новые методы построения функциональных наблюдателей состояния для линейных положительных систем дробного порядка с неизвестными входами. Получены условия существования таких наблюдателей, разработан вычислительный подход, базирующийся на задаче линейного программирования для определения матриц наблюдения. Также рассмотрен случай, когда в системе отсутствует запаздывание. Представлены три численных примера для иллюстрации эффективности полученных результатов.*

Ключевые слова: *системы дробного порядка, положительные системы, системы с запаздыванием, функциональные положительные наблюдатели состояния, линейное программирование, неизвестные входы.*

DESIGN OF LINEAR FUNCTIONAL STATE OBSERVERS FOR LINEAR POSITIVE FRACTIONAL-ORDER TIME-DELAY SYSTEMS WITH UNKNOWN INPUTS

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This paper addresses the problem of designing positive functional observers for linear fractional-order time-delay positive systems with unknown inputs. The proposed functional observers are positive, that is, they ensure that the estimates are nonnegative at any time. Moreover, necessary and sufficient conditions for the existence of such positive functional observers are formulated in terms of linear programming.

The purpose of the article is to design positive functional observers for linear fractional-order time-delay systems with unknown inputs.

Material and methods. *Functional observers for linear fractional-order time-delay system were research materials. Methods of mathematical analysis, linear algebra and numerical simulation were used in the research.*

Findings and their discussion. *Two theorems are proved. Theorem 1 asserts that the observer we consider is a positive linear functional one under some conditions. Theorem 2 states that the observer gains observer matrices if corresponding LP problem is feasible. Three numerical examples are provided to demonstrate the effectiveness of obtained results.*

Conclusion. *New results for designing positive functional state observers for linear positive time-delay systems with unknown inputs are proposed. Conditions for the existence of positive functional observers are derived and computational approach based on LP is given for the determination of the observer matrices. The case where there is no time delay in the system is also discussed. Three numerical examples are given to illustrate the effectiveness of the proposed design method.*

Key words: *Fractional-order systems, positive systems, time-delay systems, functional positive observers, linear programming, unknown inputs.*

1 Introduction

Positive systems have numerous applications in science and engineering. They are used as models for a variety of phenomena in the life sciences, physics and technology, chemistry, economics, populations dynamics and ecology. A dynamical system is called positive if for any nonnegative initial condition, the corresponding solution of the system is also nonnegative. Positive dynamical systems play an important role in the modelling of dynamical phenomena whose variables are restricted to be nonnegative (see, for example,[1−7]).

In the past few decades, many authors pointed out that several areas of physics, control engineering and signal processing may be precisely described with the help of fractional calculus. A large number of monographs and papers have been devoted to fractional dynamical systems (see, for example, [8−38]).

Positive linear systems with different fractional orders have been addressed in the literature. In particular, Nigmatullin and Nelson described in terms of fractional kinetics in complex systems [14]; Jesus, Machado and Cunha analyzed the fractional-order dynamics in botanical electrical impedances [15]; Petrovic, Spasic and Atanackovic developed a fractional-order mathematical model of a human root dentin [16]. Fractional ordinary differential equations are naturally related to systems with memory which exists in most biological systems. Also,

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they are closely related to fractals, which are abundant in biological systems [21]. It has been deduced that the membranes of cells of biological organism have fractional-order electrical conductance and then are classified in groups of non-integer order models. Fractional derivatives embody essential features of cell rheological behavior and have enjoyed greatest success in the field of rheology. Also, it has been shown that modelling the behavior of brainstem vestibule-oculumotor neurons by fractional ordinary differential equations has more advantages than classical integer-order modelling [23].

In many practical systems, all the state variables are not available for measurements, although some components of the state vector have to be known for control strategies. So, one can find a functional state observer, that estimates a linear combination of the states of a system using the input and output measurements. The problem of design of state observers on integer-order systems have been reported in [39−45]. For fractional-order systems, some interesting results on the problem were reported in [46−49]. However, research into state observer design of fractional linear systems with time-delay has, so far, received lesser attention. Recently, in [50], by using the fractional-order Lyapunov approach, the authors proposed the functional observers design for linear fractional-order time-delay systems. To the best of our knowledge, there is no work in which the problem of designing fractional order observer for time-delay positive fractional-order systems in which unknown inputs are investigated.

In this paper, we consider the problem of designing positive linear functional observers for time-delay positive fractional-order systems with unknown inputs. Our proposed observers are positive, that is, they ensure that the estimates are nonnegative at any time. We derive new conditions for the existence of such positive linear functional observers. We propose a computational approach based on linear programming (LP) for the determination of the observer's parameters.

The main contribution of this paper is that, for the first time, the problem of designing positive linear functional observers for time-delay positive systems with unknown inputs and fractional-order $\alpha \in (0, 1]$ has been considered. As a particular case when order $\alpha = 1$, our result can be considered as an extension result of [43−45].

This paper is organized as follows. In section 2, we provide the preliminaries. The main results are given in Section 3. In section 4, we provide three numerical examples to demonstrate the effectiveness of our obtained results. Finally, a conclusion is drawn in Section 5.

Notations: *I_n denotes the $n \times n$ identity matrix, $0_{m,n}$ denotes the $m \times n$ zero matrix. $\mathbb{R}^n_+$ denotes the nonnegative orthant of the n-dimensional real space $\mathbb{R}^n$. For a vector $x = (x_i) \in \mathbb{R}^n$, we denote $|x| = (|x_i|) \in \mathbb{R}^n_+$. For a real matrix M, M^T denotes the transpose, $M \geq 0$ is called a nonnegative matrix if all of its components are nonnegative (i.e. $m_{ij} \geq 0$ for all i, j).*

The Caputo derivative of function $f(t)$ with order $\alpha \in (0, 1]$ is defined by

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \dot{f}(\tau) t\tau, $$

where $\dot{f}$ is the first order derivative of function f and the function $\Gamma(\cdot)$ is defined as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \ z \in \mathbb{R}.$$

2 Preliminaries

This section presents some definitions and preliminary results which will be used throughout the paper.

Consider the following linear fractional-order time-delay system

$$\begin{aligned} {}_0^C D_t^\alpha x(t) = Ax(t) + \sum_{i=1}^{q} A_i x(t-\tau_i) \\ + Bu(t) + Ff(t), \ t \geq 0, \end{aligned} \tag{1}$$

$$x(\theta) = \phi(\theta), \ \forall \theta \in [-\tau, 0], \ \tau = \max_{1 \leq i \leq q} \tau_i, \tag{2}$$

$$y(t) = Cx(t), \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $y(t) \in \mathbb{R}^p$ is the output vector, $f(t) \in \mathbb{R}^{n_f}$ is the disturbance vector. By ${}_0^C D_t^\alpha x(t)$ we mean that ${}_0^C D_t^\alpha x(t) = \left[{}_0^C D_t^\alpha x_1(t) \, {}_0^C D_t^\alpha x_2(t) \ldots {}_0^C D_t^\alpha x_n(t)\right]^T$, $\alpha \in (0,1]$. $A \in \mathbb{R}^{n \times n}$, $A_i \in \mathbb{R}^{n \times n}$ $(i = 1, 2, \ldots, q)$, $B \in \mathbb{R}^{n \times m}$, $F \in \mathbb{R}^{n \times n_f}$ and $C \in \mathbb{R}^{p \times n}$ are constant matrices, the time delay τ_i $(i = 1, 2, \ldots, q)$ is assumed to be known positive constants, and $\phi(\theta) \in \mathbb{R}^n$ is the initial function.

Let us state the following definition of positive system (1)-(2).

Definition 1. *The linear time-delay system* (1)-(2) *is said to be positive if for any nonnegative initial condition $\phi(\theta) \in \mathbb{R}_+^n$, $\forall \theta \in [-\tau, 0]$ and any input $u(t) \in \mathbb{R}_+^m$, $\forall t \geq 0$, the corresponding trajectory $x(t) \in \mathbb{R}_+^n$ for all $t \geq 0$.*

The following definition will be used in this paper.

Definition 2 ([1]). *A square real matrix M is called a Metzler matrix if its off-diagonal elements are nonnegative, i.e. $m_{ij} \geq 0$, $i \neq j$.*

In the subsequent, we let $\phi(\theta) \in \mathbb{R}_+^n$, $\forall \theta \in [-\tau, 0]$, $u(t) \in \mathbb{R}_+^m$, $\forall t \geq 0$ and $f(t) \in \mathbb{R}_+^{n_f}$, $\forall t \geq 0$. Then from [32], we obtain the following condition which ensures the positivity of system (1)-(2).

Lemma 1. *System* (1)-(2) *is positive if and only if A is a Metzler matrix, A_i $(i = 1, 2, \ldots, q)$, B and F are nonnegative matrices.*

Let us now assume that the system defined in (1)-(2) is positive. Let $z(t) = Lx(t) \in \mathbb{R}_+^r$, $1 \leq r \leq n$, be defined as a positive linear function of the state vector, where $L \geq 0$ is any given $r \times n$ matrix. Our objective in this paper is to design a positive linear functional observer $\hat{z}(t) \in \mathbb{R}_+^r$ such that the error

$e(t) = z(t) - \hat{z}(t)$ converges asymptotically to zero as $t \to \infty$. To achieve the objective, we consider the following rth-order positive linear functional observer

$$\hat{z}(t) = \omega(t) + Ey(t), \tag{4}$$

$$ {}^C_0D^\alpha_t \omega(t) = N\omega(t) + \sum_{i=1}^{q} N_i\omega(t-\tau_i) + Jy(t) + \sum_{i=1}^{q} J_i y(t-\tau_i) + Hu(t),\ t \geq 0, \tag{5}$$

$$\omega(\theta) = \rho(\theta) \in \mathbb{R}^r_+,\ \forall\theta \in [-\tau, 0], \tag{6}$$

where $E \in \mathbb{R}^{r\times p}$, $N \in \mathbb{R}^{r\times r}$, $J \in \mathbb{R}^{r\times p}$, $N_i \in \mathbb{R}^{r\times r}$ and $J_i \in \mathbb{R}^{r\times p}$ $(i = 1, 2, \ldots, q)$ are matrices to be determined.

Definition 3. *The observer defined in* (4)-(6) *is called a positive linear functional observer of system* (1)-(3) *if for any initial condition,* $\rho(\theta) \in \mathbb{R}^r_+,\ \forall\theta \in [-\tau, 0]$, *all inputs* $u(t) \in \mathbb{R}^m_+,\ \forall t \geq 0$ *and all disturbances* $f(t) \in \mathbb{R}^{n_f}_+,\ \forall t \geq 0$, *then* $\hat{z}(t) \in \mathbb{R}^r_+$ *for all* $t \geq 0$ *and* $\hat{z}(t)$ *converges asymptotically to* $z(t)$ *as* $t \to \infty$.

3 Main results

The following theorem provides conditions which ensure that (4)-(5) is a positive linear functional observer of system (1)-(3).

Theorem 1. *The observer defined in* (4)-(5) *is a positive linear functional observer of system* (1)-(3) *if the following conditions are satisfied for* $i = 1, 2, \ldots, q$

$$N \ is \ Metzler,\ N_i \geq 0, \tag{7}$$

$$JC \geq 0,\ J_iC \geq 0, \tag{8}$$

$$ {}^C_0D^\alpha_t e(t) = Ne(t) + \sum_{i=1}^{q} N_i e(t-\tau_i) \ \ is \ asymptotically \ stable, \tag{9}$$

$$NL + JC + ECA - LA - NEC = 0, \tag{10}$$

$$N_iL + J_iC + ECA_i - LA_i - N_iEC = 0, \tag{11}$$

$$EC \geq 0,\ ECB \leq LB, \tag{12}$$

$$ECF - LF = 0, \tag{13}$$

$$H - LB + ECB = 0. \tag{14}$$

Proof. We first prove that $\hat{z}(t) \in \mathbb{R}_+^r$ for all $t \geq 0$ if conditions (7)-(8) and (12) are satisfied. For this, let us consider the following augmented system

$$\begin{aligned}\begin{bmatrix} {}_0^C D_t^\alpha x(t) \\ {}_0^C D_t^\alpha \omega(t) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ JC & N \end{bmatrix} \begin{bmatrix} x(t) \\ \omega(t) \end{bmatrix} \\ &+ \sum_{i=1}^{q} \begin{bmatrix} A_i & 0 \\ J_iC & N_i \end{bmatrix} \begin{bmatrix} x(t-\tau_i) \\ \omega(t-\tau_i) \end{bmatrix} \\ &+ \begin{bmatrix} B \\ H \end{bmatrix} u(t) + \begin{bmatrix} F \\ 0 \end{bmatrix} f(t),\ t \geq 0, \qquad (15) \\ \begin{bmatrix} x(\theta) \\ \omega(\theta) \end{bmatrix} &= \begin{bmatrix} \phi(\theta) \\ \rho(\theta) \end{bmatrix} \in \mathbb{R}_+^{n+r},\ \forall\theta \in [-\tau, 0]. \qquad (16)\end{aligned}$$

By Lemma 1, $\hat{z}(t) \in \mathbb{R}_+^r$ for all $t \geq 0$ if $EC \geq 0$, $ECB \leqslant LB$, $\begin{bmatrix} A & 0 \\ JC & N \end{bmatrix}$ is Metzler and $\begin{bmatrix} A_i & 0 \\ J_iC & N_i \end{bmatrix} \geq 0$, which is equivalent to conditions (7)-(8) and (12), since (1)-(2) is a positive system.

Next, we prove that (9)-(11), (13)-(14) are sufficient conditions that ensure $\hat{z}(t)$ converges asymptotically to $z(t)$ as $t \to \infty$. In regard to (1) and (4)-(5), the dynamic of the fractional-order error is given by

$$\begin{aligned}{}_0^C D_t^\alpha e(t) &= {}_0^C D_t^\alpha z(t) - {}_0^C D_t^\alpha \hat{z}(t) \\ &= Ne(t) + \sum_{i=1}^{q} N_i e(t-\tau_i) \\ &\quad + (LA + NEC - NL - ECA - JC)x(t) \\ &\quad + \sum_{i=1}^{q} (LA_i + N_iEC - N_iL \\ &\quad - ECA_i - J_iC)x(t-\tau_i) \\ &\quad + (LB - ECB - H)u(t) \\ &\quad + (LF - ECF)f(t),\ t \geq 0, \qquad (17) \\ e(\theta) &= L\phi(\theta) - \rho(\theta),\ \theta \in [-\tau, 0]. \qquad (18)\end{aligned}$$

It is clear from (17) that $e(t) \to 0$ as $t \to \infty$ if conditions (9)-(11), (13) and (14) of Theorem 1 are satisfied. Hence, $\hat{z}(t)$ converges asymptotically to $z(t)$ as $t \to \infty$. This completes the proof of Theorem 1. ■

Remark 1. From Theorem 1, the design of a positive linear functional observer now rest with determining unknown observer parameters E, N, J, N_i and J_i $(i = 1, 2, \ldots, q)$ such that conditions (7)-(14) of Theorem 1 are satisfied. For this, we first solve the unknown matrix E such that the conditions (12)-(13) hold. And then, we will solve the remaining unknown matrices N, J, N_i and J_i $(i = 1, 2, \ldots, q)$ such that the conditions (7)-(11) are satisfied.

It follows from (13) that a solution for E always exists if and only if rank $\begin{bmatrix} CF \\ LF \end{bmatrix}$ = rank$[CF]$. Under this condition, a general solution for E is given by

$$E = (LF)(CF)^+ + S[I_p - (CF)(CF)^+], \tag{19}$$

where $S \in \mathbb{R}^{n \times p}$ is arbitrary matrix. By using (19), condition (12) can be expressed as

$$(LF)(CF)^+C + S[I_p - (CF)(CF)^+]C \geq 0, \tag{20}$$

$$(LF)(CF)^+CB + S[I_p - (CF)(CF)^+]CB \leq LB. \tag{21}$$

From (20) and (21) we see that there exists matrix E such that conditions (12) and (13) of Theorem 1 can be satisfied if and only if the following LP problem in the varible $S \in \mathbb{R}^{n \times p}$ is feasible:

$$-\Psi_0^T C^T S^T \leq \Phi_0^T C^T, \tag{22}$$

$$B^T C^T \Psi_0^T S^T \leq B^T L^T - B^T C^T \Phi_0^T, \tag{23}$$

where $\Phi_0 = (LF)(CF)^+$ and $\Psi_0 = I_p - (CF)(CF)^+$.

Next, we will determine the remaining unknown matrices N, J, N_i and J_i $(i = 1, 2, \ldots, q)$ such that the conditions (7)-(11) of Theorem 1 are satisfied.

By extending Theorem 2 in [32], we obtain the following condition which is equivalent to (6) and (8).

Lemma 2. *Let N be a Metzler matrix and $N_i \geq 0$ $(i = 1, 2, \ldots, q)$, then system*

$$\begin{cases} {}_0^C D_t^\alpha e(t) = Ne(t) + \sum_{i=1}^q N_i e(t - \tau_i) \\ e(\theta) = \varphi(\theta), \ \theta \in [-\tau, 0] \end{cases} \tag{24}$$

is asymptotically stable if and only if $N + \sum_{i=1}^q N_i$ is Hurwitz.

Remark 2. By Lemma 2, it follows that the positive system (24) is asymptotically stable if and only if there exists a vector $\lambda > 0$ such that

$$(N + \sum_{i=1}^q N_i)^T \lambda < 0. \tag{25}$$

Provided that matrix E is solved from LP problem (22)-(23), we will solve equations (7)-(11). For this, we represent these equations into the following form

$$\chi X = Y, \tag{26}$$

where

$$\chi = \begin{bmatrix} N \ J \ N_1 \ J_1 \ \ldots \ N_q \ J_q \end{bmatrix}, \tag{27}$$

$$X = \text{block-diag}\left(\begin{bmatrix} K \\ C \end{bmatrix}, \begin{bmatrix} K \\ C \end{bmatrix}, \ldots, \begin{bmatrix} K \\ C \end{bmatrix} \right),$$

$$X \in \mathbb{R}^{(q+1)(p+r) \times (q+1)n}, \tag{28}$$

$$Y = \begin{bmatrix} M \ M_1 \ M_2 \ \ldots \ M_q \end{bmatrix} \in \mathbb{R}^{r \times (q+1)n}. \tag{29}$$

In (28)-(29), $K = L - EC$, $M = LA - ECA$, $M_i = LA_i - ECA_i$ for $i = 1, 2, \ldots, q$.

Since X and Y are two known constant matrices, a solution for χ always exists if and only if

$$\operatorname{rank} \begin{bmatrix} X \\ Y \end{bmatrix} = \operatorname{rank}(X). \tag{30}$$

Under condition (30), a general solution for χ is given by

$$\chi = YX^+ + Z\left(I_{(q+1)(p+r)} - XX^+\right), \tag{31}$$

where $X^+ \in \mathbb{R}^{(q+1)n \times (q+1)(p+r)}$ is the Moor-Penrose inverse of X and $Z \in \mathbb{R}^{r \times (q+1)(p+r)}$ is an arbitrary matrix to be determined. Moreover, matrices N, J, N_i, J_i $(i = 1, 2, \ldots, q)$ can now be extracted from (31) and are expressed as

$$N = \Phi e_N + Z\Psi e_N, \tag{32}$$
$$J = \Phi e_J + Z\Psi e_J, \tag{33}$$
$$N_i = \Phi e_{N_i} + Z\Psi e_{N_i}, \tag{34}$$
$$J_i = \Phi e_{J_i} + Z\Psi e_{J_i}, \tag{35}$$

where

$$\Phi = YX^+,\ \Psi = I_{(q+1)(p+r)} - XX^+ \tag{36}$$

and $e_N, e_{N_i} \in \mathbb{R}^{(q+1)(p+r) \times r}$, $e_J, e_{J_i} \in \mathbb{R}^{(q+1)(p+r) \times p}$ are the following

$$e_N = \begin{bmatrix} I_r \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, e_{N_i} = \begin{bmatrix} 0 \\ 0 \\ \overbrace{I_r}^{i+2-th} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$e_J = \begin{bmatrix} 0 \\ I_p \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},\ e_{J_i} = \begin{bmatrix} 0 \\ 0 \\ \overbrace{I_p}^{i+3-th} \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ i = 1, \ldots, q. \tag{37}$$

Now, in order to implement a positive functional observer (4)-(6), we will gather conditions (7)-(11) to formulate an LP-based problem for checking the design parameters. By using (32)-(35), conditions (25) can be represented as

$$(e_N^T + \sum_{i=1}^{q} e_{N_i}^T)\Phi^T \lambda + (e_N^T + \sum_{i=1}^{q} e_{N_i}^T)\Psi^T \Gamma \mathbf{1}_r < 0. \tag{38}$$

On the other hand, conditions $JC \geqslant 0$ and $J_iC \geqslant 0$ $(i = 1, 2, \ldots, q)$ are equivalent to the following, respectively,

$$C^T e_J^T \Phi^T \lambda + C^T e_J^T \Psi^T \Gamma \mathbf{1}_r \geqslant 0 \tag{39}$$

and

$$C^T e_{J_i}^T \Phi^T \lambda + C^T e_{J_i}^T \Psi^T \Gamma \mathbf{1}_r \geqslant 0,\ i = 1, \ldots, q. \tag{40}$$

Based the above discussion we obtain the following theorem which provides a computational approach which is based on LP for the determination of the parameters N, J, N_i, J_i $(i = 1, 2, \ldots, q)$ of positive linear functional observers.

Theorem 2. *If the following LP problem in the variables* $\lambda \in \mathbb{R}^r$ *and* $\Gamma \in \mathbb{R}^{(q+1)(p+r)\times r}$ *is feasible:*

$$\begin{cases} \lambda > 0, \\ (e_N^T + \sum_{i=1}^q e_{N_i}^T)\Phi^T \lambda + (e_N^T + \sum_{i=1}^q e_{N_i}^T)\Psi^T \Gamma \mathbf{1}_r < 0, \\ C^T e_J^T \Phi^T \lambda + C^T e_J^T \Psi^T \Gamma \mathbf{1}_r \geqslant 0, \\ C^T e_{J_i}^T \Phi^T \lambda + C^T e_{J_i}^T \Psi^T \Gamma \mathbf{1}_r \geqslant 0,\ i = 1, \ldots, q, \end{cases} \tag{41}$$

then the observer gains N, J, N_i, J_i $(i = 1, 2, \ldots, q)$ *are obtained as in (32)-(35) where* $Z = (diag(\lambda))^{-1}\Gamma^T$.

Remark 3. Positive functional observers without internal delay. Here, we can easily derive a computational approach based on LP to derive the following positive functional observer for system (1)-(2)

$$\hat{z}(t) = \omega(t) + Ey(t), \tag{42}$$

$$\begin{aligned} {}_0^C D_t^\alpha \omega(t) &= N\omega(t) + Jy(t) + \sum_{i=1}^q J_i y(t - \tau_i) \\ &\quad + Hu(t),\ t \geq 0, \end{aligned} \tag{43}$$

$$\omega(\theta) = \rho(\theta) \in \mathbb{R}_+^r,\ \forall \theta \in [-\tau, 0], \tag{44}$$

where $N \in \mathbb{R}^{r\times r}$, $J \in \mathbb{R}^{r\times p}$ and $J_i \in \mathbb{R}^{r\times p}$ $(i = 1, 2, \ldots, q)$. The following proposition provides a computational approach for determining the matrices N, J and J_i $(i = 1, 2, \ldots, q)$.

Proposition 1. *If the following LP problem in the variables* $\lambda \in \mathbb{R}^r$ *and* $\Gamma \in \mathbb{R}^{(p(q+1)+r)\times r}$ *is feasible:*

$$\begin{cases} \lambda > 0, \\ e_N^T \Phi^T \lambda + e_N^T \Psi^T \Gamma \mathbf{1}_r < 0, \\ C^T e_J^T \Phi^T \lambda + C^T e_J^T \Psi^T \Gamma \mathbf{1}_r \geqslant 0, \\ C^T e_{J_i}^T \Phi^T \lambda + C^T e_{J_i}^T \Psi^T \Gamma \mathbf{1}_r \geqslant 0,\ i = 1, \ldots, q, \end{cases} \tag{45}$$

then the observer gains N, J, J_i $(i = 1, 2, \ldots, q)$ *are obtained as below:*

$$N = \Phi e_N + Z\Psi e_N, \tag{46}$$

$$J = \Phi e_J + Z\Psi e_J, \tag{47}$$

$$J_i = \Phi e_{J_i} + Z\Psi e_{J_i}, \tag{48}$$

where

$$\Phi = YX^{+},\ \Psi = I_{(p(q+1)+r)} - XX^{+}, \tag{49}$$

$Z = (diag(\lambda))^{-1}\Gamma^{T}$, *Y is defined as in* (25) *and* $X \in \mathbb{R}^{(p(q+1)+r)\times(q+1)n}$, $e_N \in \mathbb{R}^{(p(q+1)+r)\times r}$, $e_J, e_{J_i} \in \mathbb{R}^{(p(q+1)+r)\times p}$ *are the following*

$$e_N = \begin{bmatrix} I_r \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix},\ e_J = \begin{bmatrix} 0 \\ I_p \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

$$e_{J_i} = \begin{bmatrix} 0 \\ 0 \\ \overbrace{I_p}^{i+2-th} \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ i = 1, \ldots, q. \tag{50}$$

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix},\ X_1 = \begin{bmatrix} K\ 0 \end{bmatrix},$$
$$X_2 = block\text{-}diag\left(C, C, \ldots, C\right). \tag{51}$$

Remark 4. For the case where $A_i = 0,\ i = 1, 2, \ldots, q$ i.e. system (1)-(3) has no time delay, the following result is an extension of the work [43] to the design of reduced-order positive functional observers. Note that in [43], only full-order Luenberger positive observers for linear positive systems with no inputs were considered. Now, we have the following reduced-order positive functional observer

$$\hat{z}(t) = \omega(t) + Ey(t), \tag{52}$$

$${}_0^C D_t^{\alpha}\omega(t) = N\omega(t) + Jy(t) + Hu(t),\ t \geq 0, \tag{53}$$

$$\omega(0) = \omega_0, \tag{54}$$

where $\hat{z}(t)$ is the estimate of $z(t) = Fx(t)$, $\hat{z}_0 \in \mathbb{R}_+^r$ is the initial condition and $F \geq 0$. We can obtain sufficient conditions to ensure that $\hat{z}(t) \in \mathbb{R}_+^r$ and $\hat{z}(t)$ converges asymptotically to $z(t)$ as $t \to \infty$ for any $\hat{z}_0 \in \mathbb{R}_+^r$ and $u(t) \in \mathbb{R}_+^m$. The conditions are as given below

$$\begin{cases} N \text{ is Metzler and Hurwitz}, \\ JC \geq 0, \\ NF + JC - FA = 0, \\ EC \geq 0,\ ECB \leq LB, \\ ECF - LF = 0, \\ H - LB + ECB = 0. \end{cases} \tag{55}$$

Hence, N and J can be determined if the following LP problem in the variables $\lambda \in \mathbb{R}^r$ and $\Gamma \in \mathbb{R}^{(p+r)\times r}$ is feasible

$$\begin{cases} \lambda > 0, \\ e_N^T \Phi^T \lambda + e_N^T \Psi^T \Gamma \mathbf{1}_r < 0, \\ C^T e_J^T \Phi^T \lambda + C^T e_J^T \Psi^T \Gamma \mathbf{1}_r \geqslant 0, \end{cases} \tag{56}$$

then the observer gains N, J are obtained as below:

$$N = \Phi e_N + Z\Psi e_N, \tag{57}$$

$$J = \Phi e_J + Z\Psi e_J, \tag{58}$$

where

$$\Phi = YX^+,\ \Psi = I_{p+r} - XX^+, \tag{59}$$

$Z = (\mathrm{diag}(\lambda))^{-1}\Gamma^T$ and $X = \begin{bmatrix} K \\ C \end{bmatrix} \in \mathbb{R}^{(p+r)\times n}$, $Y = M$, $e_N = \begin{bmatrix} I_r \\ 0 \end{bmatrix} \in \mathbb{R}^{r+p)\times r}$, $e_J = \begin{bmatrix} 0 \\ I_p \end{bmatrix} \in \mathbb{R}^{(r+p)\times p}$.

Algorithm 1
Step 1: *Check if* rank $\begin{bmatrix} CF \\ LF \end{bmatrix}$ = rank$[CF]$. *If so, solve the LP (41) to obtain matrix E.*
Step 2: *Obtain matrices X and Y from* (28)-(29). *Check the existence condition* (30).
Step 3: *Compute the matrices* Φ *and* Ψ *from* (36).
Step 4: *Solve the LP (41) with respect to* Γ *and* λ.
Step 5: *Compute the matrix* $Z = (diag(\lambda))^{-1}\Gamma^T$ *where* (λ, Γ) *is a solution obtained in Step 4.*
Step 6: *By substituting Z into* (32)-(35) *to obtain observer gains* N, N_i, J, J_i $(i = 1, 2, \ldots, q)$.

4 Numerical examples

Example 1. Let us first consider an example, where a positive 3th-order system with one output and matrices A, A_1, B, C, F and L are as given below

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -3 & 0 \\ 0 & 1 & -5 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.1 & 0.1 & 0.2 \\ 0.1 & 0 & 0 \\ 0.2 & 0.1 & 0.05 \end{bmatrix},\ F = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^\top,\ L = \begin{bmatrix} 0.25 \\ 1 \\ 0.5 \end{bmatrix}^\top.$$

Clearly, this system is a positive system since A is a Metzler and Hurwitz and $A_1 \geq 0$, $B \geq 0$, $C \geq 0$, $F \geq 0$, $L \geq 0$.

According to Step 1 of the Algorithm 1, we obtain $E = 0.25$.

According to steps (Step 2-Step 6) of the Algorithm 1, we obtain $N = -5$, $N_1 = 0.05$, $J = \begin{bmatrix} 1 & 2.5 \end{bmatrix}$, $J_1 = \begin{bmatrix} 0.2 & 0 \end{bmatrix}$. With N, N_1, J and J_1 as given, it is easy to check that all the conditions of the Theorem 1 are satisfied. This completes the design of a first-order positive observer to estimate $z(t) = 0.25x_1(t) + x_2(t) + 0.5x_3(t)$.

Simulation results

To show the simulation results, we consider the input $u(t) = 0.001e^t$, $0 \leqslant t \leqslant 10$, $\tau = 1s$ and the initial conditions are $x_1(\theta) = 1$, $x_2(\theta) = 2$, $x_3(\theta) = 3$, $w(\theta) = 0$ for $\theta \in [-\tau, 0]$. Fig. 1 shows the performance of the functional observer presented in this paper for $\alpha = 0.75$ the unknown input $f(t) = 1 + \sin t$, $0 \leqslant t \leqslant 10$.

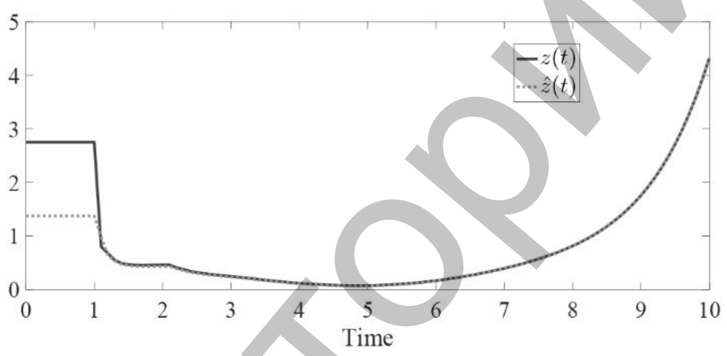

Fig. 1: State response $z(t)$ and its estimate $\hat{z}(t)$ in Example 1 with fractional order $\alpha = 0.75$

Example 2. Let us next consider an example, where a positive 4th-order system with two outputs and matrices A, A_1, B, C, F and L are as given below

$$A = \begin{bmatrix} -8 & 0 & 0 & 0 \\ 1 & -4 & 0 & 1 \\ 8 & 0 & -3 & 2 \\ 6 & 0 & 1 & -9 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \ F = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.3162 & 0 & 0 & 0 \\ 1.8974 & 0.6325 & 0.9487 & 1.2649 \\ 0.6325 & 0 & 0.3162 & 0 \\ 0.9487 & 0 & 1.5811 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \ L = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

Clearly, this system is a positive system since A is a Metzler and Hurwitz and $A_1 \geq 0$, $B \geq 0$, $C \geq 0$, $F \geq 0$, $L \geq 0$.

According to Step 1 of the Algorithm, we obtain $E = \begin{bmatrix} 4 \; 3 \end{bmatrix}^\top$.

According to steps (Step 2-Step 6) of the Algorithm 1, we obtain $N = \begin{bmatrix} -3 & 2 \\ 1 & -9 \end{bmatrix}$, $N_1 = \begin{bmatrix} 0.3162 & 0 \\ 1.5811 & 0 \end{bmatrix}$, $J = \begin{bmatrix} 25 \\ 8 \end{bmatrix}$, $J_1 = \begin{bmatrix} 0.6325 \\ 5.3759 \end{bmatrix}$. With N, N_1, J and J_1 as given, it is easy to check that all the conditions of the Theorem 1 are satisfied. This completes the design of a second-order positive observer to estimate $z_1(t) = x_1(t) + x_3(t)$ and $z_2(t) = 2x_1(t) + x_4(t)$.

Simulation results

To show the simulation results, we consider the input $u(t) = 0.001e^{0.01t}$, $0 \leqslant t \leqslant 20$, $\tau = 1s$ and the initial conditions are $x_1(\theta) = 1$, $x_2(\theta) = 2$, $x_3(\theta) = 3$, $x_4(\theta) = 4$, $w_1(\theta) = 0$, $w_2(\theta) = 0$ for $\theta \in [-\tau, 0]$. Fig. 2 anf Fig.3 show the performance of the functional observer presented in this paper for $\alpha = 0.81$ the unknown input $f(t) = 1 + \sin t$, $0 \leqslant t \leqslant 20$.

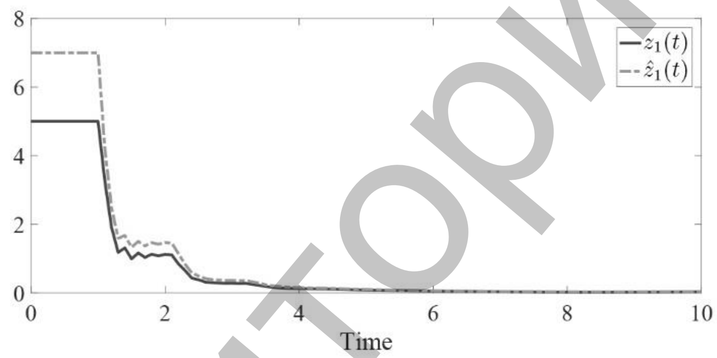

Fig. 2: State response $z_1(t)$ and its estimate $\hat{z}_1(t)$ in Example 2 with fractional order $\alpha = 0.81$

Example 3. Let us first consider an example, where a positive 3th-order system without delay and matrices A, B, C, F and L are as given below

$$A = \begin{bmatrix} -0.5 & 0.1 & 0.1 \\ 0.1 & 0 & 0 \\ 0 & 0.1 & -0.9 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \; F = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; L = \begin{bmatrix} 0.5 \; 1 \; 1 \end{bmatrix}.$$

Clearly, this system is a positive system since A is a Metzler and Hurwitz and $B \geq 0$, $C \geq 0$, $F \geq 0$, $L \geq 0$.

According to Step 1 of the Algorithm 1, we obtain $E = 0.5$.

According to steps (Step 2-Step 6) of the Algorithm 1, we obtain : $N = -0.9$, $J = \begin{bmatrix} 0.1 \; 1 \end{bmatrix}$. With N and J as given, it is easy to check that all the conditions

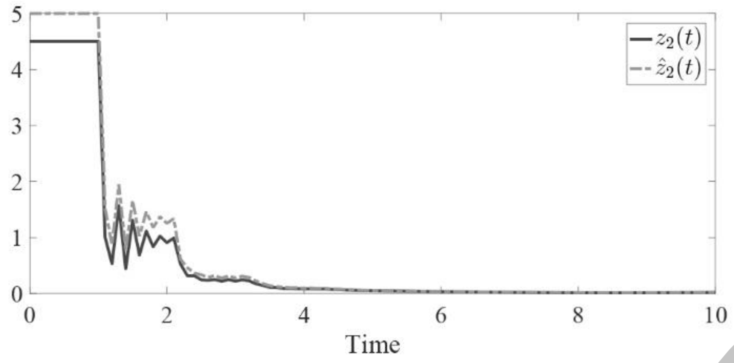

Fig. 3: State response $z_2(t)$ and its estimate $\hat{z}_2(t)$ in Example 2 with fractional order $\alpha = 0.81$

of the Theorem 1 are satisfied. This completes the design of a fractional-order positive observer to estimate $z(t) = 0.5x_1(t) + x_2(t) + x_3(t)$.

Simulation results

To show the simulation results, we consider the input $u(t) = 0.001e^t,\ 0 \leqslant t \leqslant 7$ and the initial conditions are $x_1(0) = 1,\ x_2(0) = 2,\ x_3(0) = 3,\ w(0) = 0$. Fig. 3 shows the performance of the functional observer presented in this paper for $\alpha = 0.5$ the unknown input $f(t) = 1 + \sin t,\ 0 \leqslant t \leqslant 7$.

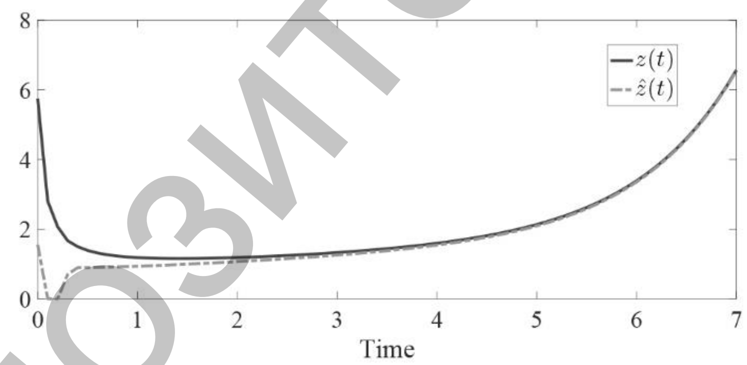

Fig. 4: State response $z(t)$ and its estimate $\hat{z}(t)$ in Example 3 with fractional order $\alpha = 0.5$

5 Conclusion

In this paper, we have proposed new results for designing positive functional state observers for linear positive time-delay systems with unkown ipputs. Conditions for the existence of positive functional observers have been derived and a computational approach based on LP has been given for the determination of the observer matrices. The case where there is no time delay in the system has also been discussed in this paper. Three numerical examples have been given to illustrate the effectiveness of the proposed design method.

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