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LATTICES OF PARTIALLY LOCAL FITTING CLASSES

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Abstract: This article deals only with finite groups. We prove the surjectivity of the mapping from the lattice of all normal Fitting classes into the lattice of the Lockett section generated by the Fitting classes that are not Lockett classes. Moreover, we find a sufficient surjectivity condition for the mapping of the lattice of the Lockett section generated by arbitrary Fitting classes into the lattice of the Lockett section generated by ω-local Fitting classes. This confirms Lockett's conjecture for the ω-local Fitting classes of a given characteristic.

Keywords: lattice of Fitting classes, ω-local Fitting class, Lockett class, Lockett section, Lockett's conjecture

Introduction

The sets of all Fitting classes and formations are complete lattices by inclusion $\subseteq$.

Recall that a *Fitting class* is a class $\mathfrak{F}$ of groups which satisfies the conditions:

(1) every normal subgroup of each group in $\mathfrak{F}$ belongs to $\mathfrak{F}$ as well;

(2) if two normal subgroups A and B of a group G belong to $\mathfrak{F}$ then so does their product AB.

Skiba was the first who applied the lattice-theoretic methods to the theory of formations of groups, proving in [1] that the lattice of all (local) formations is modular. However, we lack any sufficient understanding of the lattice of Fitting classes. For instance, it is presently unknown whether the lattice of all (at least solvable) Fitting classes is modular. Therefore, series of studies aim at finding modular lattices of Fitting classes (see [2, Problem 14.47]). Lausch proved the modularity of the lattice of all solvable normal Fitting classes in [3]; extending this result, Bryce and Cossey proved in [4] that the lattice of Fitting classes in the Lockett section is modular and atomic.

Recall that the *Lockett section* of a Fitting class $\mathfrak{F}$, denoted [5] by $\mathrm{Locksec}(\mathfrak{F})$, is the collection of all Fitting classes $\mathfrak{H}$ for which $\mathfrak{F}^* = \mathfrak{H}^*$, where $\mathfrak{F}^*$ is defined as the smallest Fitting class including $\mathfrak{F}$ such that $(G \times H)_{\mathfrak{F}^*} = G_{\mathfrak{F}^*} \times H_{\mathfrak{F}^*}$ for all groups G and H. A Fitting class $\mathfrak{F}$ is called a *Lockett class* whenever $\mathfrak{F} = \mathfrak{F}^*$.

Recall also that a nontrivial Fitting class $\mathfrak{F}$ is called *normal* whenever given a group G its $\mathfrak{F}$-radical $G_{\mathfrak{F}}$ is an $\mathfrak{F}$-maximal subgroup of G.

Following [6], given a pair of Fitting classes $\mathfrak{F} \subseteq \mathfrak{H}$ define the mapping

$$\mathfrak{X} \to \mathfrak{X} \cap \mathfrak{F}^* \tag{1}$$

from $\mathrm{Locksec}(\mathfrak{H})$ into $\mathrm{Locksec}(\mathfrak{F})$. Given $\mathfrak{S} \subseteq \mathfrak{E}$, where $\mathfrak{S}$ and $\mathfrak{E}$ are the classes of all finite solvable groups and all finite groups respectively, (1) is a surjective mapping (see [6, X, 6.1]); in other words, the Lockett section of $\mathfrak{S}$ is defined by the Lockett section of $\mathfrak{E}$. Lockett [5] posed the problem: Is it true that (1) is always surjective provided that $\mathfrak{H} = \mathfrak{S}$? Subsequently this problem became known as "Lockett's conjecture" [5].

It is worth noting that surjective mappings were initially constructed from the lattice of all normal Fitting classes into the lattice of the Lockett section generated by the following particular cases of a local Fitting class: the hereditary class [4]; the classes of the form $\mathfrak{X}\mathfrak{N}$ and $\mathfrak{X}\mathfrak{S}_\pi\mathfrak{S}_{\pi'}$ [7]; the classes with a constant H-function, i.e., those of the form $\mathfrak{X}(\bigcap_{i\in I}\mathfrak{S}_{\pi_i}\mathfrak{S}_{\pi_i'})$ [6, X, 6.10]. For an arbitrary local Fitting class Vorob′ëv constructed the mapping in question [8].

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In this regard it is a meaningful problem to find the nonlocal Fitting classes satisfying Lockett's conjecture; i.e., the nonlocal Fitting classes the lattice of whose Lockett sections maps surjectively onto the lattice of all normal Fitting classes. We solve this problem for p-local Fitting classes.

Meanwhile, Berger and Cossey [9] constructed an example of a nonlocal Fitting class that fails to satisfy Lockett's conjecture (see [6, X, 6.16] for instance). Apart from the Berger–Cossey example [9], no Fitting classes are known yet for which Lockett's conjecture fails. Observe also that Beidleman and Hauck [7] posed the problem of finding other examples of Fitting classes failing Lockett's conjecture: Are there other nonsurjective mappings from the lattice of all normal Fitting classes into the lattice of the Lockett section generated by Fitting classes?

In this article we find new examples of nonsurjective mappings from the lattice of all normal Fitting classes into the lattice of the Lockett section generated by Fitting classes.

Recall that Doerk and Hawkes extended the original question of Lockett in [6, X, 6.1] as follows:

The Generalized Lockett Conjecture [6, X, 6.1]. *Take two Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ with $\mathfrak{F} \subseteq \mathfrak{H}$. Then $\mathfrak{F}$ satisfies Lockett's conjecture in $\mathfrak{H}$ provided that the mapping* (1) *is surjective from* Locksec($\mathfrak{H}$) *into* Locksec($\mathfrak{F}$).

In this case we refer to the Fitting class $\mathfrak{F}$ as an *$\mathscr{L}_{\mathfrak{H}}$-class.* If $\mathfrak{H} = \mathfrak{S}$ then we refer to an $\mathscr{L}_{\mathfrak{H}}$-class simply as an *$\mathscr{L}$-class.* If the same class $\mathfrak{F}$ is not an $\mathscr{L}$-class then we call it an *$\overline{\mathscr{L}}$-class.*

According to Bryce and Cossey [4] a necessary and sufficient condition for the validity of the generalized Lockett conjecture is

$$\mathfrak{F}_* = \mathfrak{F}^* \cap \mathfrak{H}_* \tag{2}$$

(see [6, X, 6.1] for instance), where the class $\mathfrak{F}_*$ is the intersection of all Fitting classes $\mathfrak{X}$ satisfying $\mathfrak{X}^* = \mathfrak{F}^*$.

Gallego [10] constructed a surjective mapping from Locksec($\mathfrak{E}$) into the lattice of the Lockett section generated by arbitrary local Fitting classes.

We prove that the mapping from Locksec($\mathfrak{X}$) of an arbitrary Fitting class $\mathfrak{X}$ into Locksec($\mathfrak{F}$) of an ω-local Fitting class $\mathfrak{F} \subseteq \mathfrak{X}$, where Char($\mathfrak{F}$) $\subseteq \omega$, is surjective. This confirms the generalized Lockett conjecture for the ω-local Fitting classes of a given characteristic.

§ 1. Preliminary Facts

We consider only finite groups and use henceforth the standard terminology (see [6, 11]).

Take $\varnothing \neq \omega \subseteq \mathbb{P}$, where $\mathbb{P}$ is the set of all primes.

Recall that the mapping $f : \omega \cup \{\omega'\} \to \{\text{Fitting classes}\}$ is called an *ω-local Hartley function* or *ω-local H-function.* Put

$$LR_\omega(f) = \{G \mid G^\omega \in f(\omega') \text{ and } F^p(G) \in f(p) \text{ for all } p \in \omega \cap \pi(G)\},$$

where $G^\omega = G^{\mathfrak{E}_\omega}$, $F^p(G) = G^{\mathfrak{N}_p \mathfrak{E}_{p'}}$, and $\mathfrak{E}_\omega$ is the class of all ω-groups. A Fitting class $\mathfrak{F}$ is called *ω-local* [11] whenever there exists an ω-local H-function f such that $\mathfrak{F} = LR_\omega(f)$.

If $\mathfrak{F} = LR_\omega(f)$, where f is an ω-local H-function, then according to the results of [11] and [12]

$$\mathfrak{F} = \Big(\bigcap_{p\in\pi_2} \mathfrak{E}_{p'}\Big) \cap \Big(\bigcap_{p\in\pi_1} f(p)\mathfrak{N}_p\mathfrak{E}_{p'}\Big) \cap f(\omega')\mathfrak{E}_\omega, \tag{3}$$

where $\pi_2 = \omega \setminus \pi_1$, $\pi_1 = \omega \cap \mathrm{Supp}(f)$, and $\mathrm{Supp}(f) = \{a \in \omega \cup \{\omega'\} \mid f(a) \neq \varnothing\}$.

Observe that in the case $\omega = \{p\}$ the Fitting class is called *p-local.*

Lemma 1 [8]. *Given a Fitting class $\mathfrak{F}$ and a saturated radical homomorph $\mathfrak{X}$, we have $(\mathfrak{F}\mathfrak{X})^* = \mathfrak{F}^*\mathfrak{X}$.*

Lemma 2 [6]. *Suppose that $\mathfrak{X}$ and $\mathfrak{Y}$ are Fitting classes. The following hold:*
(1) *If $\mathfrak{X} \subseteq \mathfrak{Y}$ then $\mathfrak{X}^* \subseteq \mathfrak{Y}^*$ and $\mathfrak{X}_* \subseteq \mathfrak{Y}_*$;*
(2) *$(\mathfrak{X}_*)_* = \mathfrak{X}_* = (\mathfrak{X}^*)_* \subseteq \mathfrak{X} \subseteq \mathfrak{X}^* = (\mathfrak{X}_*)^* = (\mathfrak{X}^*)^*$;*
(3) *$\mathfrak{X} \subseteq \mathfrak{X}_*\mathfrak{A}$;*
(4) *given a set $\{\mathfrak{F}_i \mid i \in I\}$ of nonempty Fitting classes, we have $\big(\bigcap_{i\in I} \mathfrak{F}_i\big)^* = \bigcap_{i\in I} {\mathfrak{F}_i}^*$.*

Lemma 3 [10]. *Take two Fitting classes $\mathfrak{F}$ and $\mathfrak{X}$ with $\mathfrak{F} \subseteq \mathfrak{X}$. If there exists a Fitting class $\mathfrak{Y}$ such that $(\mathfrak{F}_*\mathfrak{E}_{p'} \cap \mathfrak{F}) \vee \mathfrak{Y}\mathfrak{E}_p = \mathfrak{F}$ then $\mathfrak{X}_* \cap \mathfrak{F} \subseteq \mathfrak{F}_*\mathfrak{E}_{p'}$.*

Lemma 4 [10]. *Take a Fitting class $\mathfrak{F}$. If there exists a Fitting class $\mathfrak{Y}$ such that $\mathfrak{Y}\mathfrak{E}_p \subseteq \mathfrak{F} \subseteq \mathfrak{Y}\mathfrak{E}_p\mathfrak{E}_{p'}$ for all $p \in \mathrm{Char}(\mathfrak{F})$ then $\mathfrak{F}^* = \mathfrak{F}$.*

Lemma 5 [6]. *Given distinct primes p and q, we have $\mathfrak{S}_q\mathfrak{S}_p \not\subseteq \mathfrak{S}_*$.*

Lemma 6 [11]. *Take a Fitting class $\mathfrak{Y}$. The following are equivalent:*
(1) *$\mathfrak{Y}(F^p)\mathfrak{N}_p \subseteq \mathfrak{Y}$ for all $p \in \omega$;*
(2) *$\mathfrak{Y} = LR_\omega(f)$, where $f(\omega') = \mathfrak{Y}$ and $f(p) = \mathfrak{Y}(F^p)\mathfrak{N}_p$ for all $p \in \omega$;*
(3) *$\mathfrak{Y}$ is ω-local.*

Recall that given an arbitrary collection $\mathfrak{X}$ of groups and a prime p, we have

$$\mathfrak{X}(F^p) = \begin{cases} \mathrm{Fit}(F^p(G) \mid G \in \mathfrak{X}), & p \in \pi(\mathfrak{X}), \\ \varnothing, & p \notin \pi(\mathfrak{X}). \end{cases}$$

Lemma 7. *If $\mathfrak{X}$ and $\mathfrak{F}$ are Fitting classes then $(\mathfrak{X}_* \cap \mathfrak{F}_*)_* = (\mathfrak{X} \cap \mathfrak{F})_*$.*

Proof. Verify the inclusion $(\mathfrak{X}_* \cap \mathfrak{F}_*)_* \subseteq (\mathfrak{X} \cap \mathfrak{F})_*$. Since $\mathfrak{X}_* \subseteq \mathfrak{X}$ and $\mathfrak{F}_* \subseteq \mathfrak{F}$ by Lemma 2(2), it follows that $\mathfrak{X}_* \cap \mathfrak{F}_* \subseteq \mathfrak{X}$ and $\mathfrak{X}_* \cap \mathfrak{F}_* \subseteq \mathfrak{F}$. This implies that $\mathfrak{X}_* \cap \mathfrak{F}_* \subseteq \mathfrak{X} \cap \mathfrak{F}$. Consequently, $(\mathfrak{X}_* \cap \mathfrak{F}_*)_* \subseteq (\mathfrak{X} \cap \mathfrak{F})_*$ by Lemma 2(1).

Verify the reverse inclusion. It is obvious that $\mathfrak{X} \cap \mathfrak{F} \subseteq \mathfrak{X}$ and $\mathfrak{X} \cap \mathfrak{F} \subseteq \mathfrak{F}$. By Lemma 2(1) we have $(\mathfrak{X} \cap \mathfrak{F})_* \subseteq \mathfrak{X}_*$ and $(\mathfrak{X} \cap \mathfrak{F})_* \subseteq \mathfrak{F}_*$. Therefore, $(\mathfrak{X} \cap \mathfrak{F})_* \subseteq \mathfrak{X}_* \cap \mathfrak{F}_*$. Consequently, $((\mathfrak{X} \cap \mathfrak{F})_*)_* \subseteq (\mathfrak{X}_* \cap \mathfrak{F}_*)_*$. Lemma 2(2) yields $((\mathfrak{X} \cap \mathfrak{F})_*)_* = (\mathfrak{X} \cap \mathfrak{F})_*$.

Thus, $(\mathfrak{X}_* \cap \mathfrak{F}_*)_* = (\mathfrak{X} \cap \mathfrak{F})_*$. The proof of the lemma is complete.

In the next lemma, as well as in Sections 2 and 3, we assume all groups to be finite and solvable.

Lemma 8. *Take two Fitting classes $\mathfrak{X}$ and $\mathfrak{Y}$ such that the mapping from the lattice of all normal Fitting classes into the lattice of the Lockett section generated by $\mathfrak{X}$ is surjective, while $\mathfrak{Y}$ is a saturated radical formation. If the mapping from the lattice of all normal Fitting classes into the lattice of the Lockett section generated by $\mathfrak{X}^*\mathfrak{Y}$ is surjective then so is the mapping from the lattice of all normal Fitting classes into the lattice of the Lockett section generated by $\mathfrak{X}_*\mathfrak{Y}$.*

Proof. By [6, X, 1.19; X, 6.1] the mapping is surjective from the lattice of all normal Fitting classes into $\mathrm{Locksec}(\mathfrak{X}^*\mathfrak{Y})$ if and only if the Fitting class $\mathfrak{X}^*\mathfrak{Y}$ satisfies Lockett's conjecture: $(\mathfrak{X}^*\mathfrak{Y})_* = (\mathfrak{X}^*\mathfrak{Y})^* \cap \mathfrak{S}_*$. Since $\mathfrak{Y}$ is a saturated radical homomorph, $(\mathfrak{X}^*\mathfrak{Y})^* = (\mathfrak{X}^*)^*\mathfrak{Y}$ by Lemma 1. Lemma 2(2) yields $(\mathfrak{X}^*)^* = \mathfrak{X}^*$. Consequently,

$$(\mathfrak{X}^*\mathfrak{Y})_* = \mathfrak{X}^*\mathfrak{Y} \cap \mathfrak{S}_*. \tag{4}$$

Lemma 2(2) implies that $(\mathfrak{X}^*\mathfrak{Y})_* = ((\mathfrak{X}^*\mathfrak{Y})_*)_*$; therefore, $((\mathfrak{X}^*\mathfrak{Y})_*)_* = ((\mathfrak{X}^*\mathfrak{Y})_* \cap \mathfrak{S}_*)_*$. Verify that $\mathfrak{S}_* = (\mathfrak{S}_*\mathfrak{Y})_*$. It is obvious that $\mathfrak{S}_* \subseteq \mathfrak{S}_*\mathfrak{Y}$. Hence, by Lemma 2(1) $(\mathfrak{S}_*)_* = \mathfrak{S}_* \subseteq (\mathfrak{S}_*\mathfrak{Y})_*$. On the other hand, since $\mathfrak{Y}$ is a solvable Fitting class and $\mathfrak{S}_* \subseteq \mathfrak{S}$ by Lemma 2(2), it follows that $\mathfrak{S}_*\mathfrak{Y} \subseteq \mathfrak{S}$. Consequently, $(\mathfrak{S}_*\mathfrak{Y})_* \subseteq \mathfrak{S}_*$ by Lemma 2(1). Therefore,

$$((\mathfrak{X}^*\mathfrak{Y})_*)_* = ((\mathfrak{X}^*\mathfrak{Y})_* \cap \mathfrak{S}_*)_* = ((\mathfrak{X}^*\mathfrak{Y})_* \cap (\mathfrak{S}_*\mathfrak{Y})_*)_*.$$

Lemma 7 implies that $((\mathfrak{X}^*\mathfrak{Y})_* \cap (\mathfrak{S}_*\mathfrak{Y})_*)_* = (\mathfrak{X}^*\mathfrak{Y} \cap \mathfrak{S}_*\mathfrak{Y})_*$.

Since $\mathfrak{Y}$ is a radical formation, $\mathfrak{X}^*\mathfrak{Y} \cap \mathfrak{S}_*\mathfrak{Y} = (\mathfrak{X}^* \cap \mathfrak{S}_*)\mathfrak{Y}$. Thus,

$$(\mathfrak{X}^*\mathfrak{Y})_* = ((\mathfrak{X}^* \cap \mathfrak{S}_*)\mathfrak{Y})_*. \tag{5}$$

It follows from (4) and (5) that $((\mathfrak{X}^* \cap \mathfrak{S}_*)\mathfrak{Y})_* = \mathfrak{X}^*\mathfrak{Y} \cap \mathfrak{S}_*$.

Since the mapping from the lattice of all normal Fitting classes into the lattice of the Lockett section generated by $\mathfrak{X}$ is surjective, $\mathfrak{X}_* = \mathfrak{X}^* \cap \mathfrak{S}_*$. This implies that $(\mathfrak{X}_*\mathfrak{Y})_* = ((\mathfrak{X}^* \cap \mathfrak{S}_*)\mathfrak{Y})_*$. Therefore, $(\mathfrak{X}_*\mathfrak{Y})_* = \mathfrak{X}^*\mathfrak{Y} \cap \mathfrak{S}_*$.

Consider the Fitting class $(\mathfrak{X}_*\mathfrak{Y})^*$. Since $\mathfrak{Y}$ is a saturated radical homomorph, $(\mathfrak{X}_*\mathfrak{Y})^* = (\mathfrak{X}_*)^*\mathfrak{Y}$ by Lemma 1. Lemma 2(2) implies that $(\mathfrak{X}_*)^* = \mathfrak{X}^*$. Consequently, $(\mathfrak{X}_*\mathfrak{Y})^* = \mathfrak{X}^*\mathfrak{Y}$.

Thus, $(\mathfrak{X}_*\mathfrak{Y})_* = (\mathfrak{X}_*\mathfrak{Y})^* \cap \mathfrak{S}_*$. This means that the mapping is surjective from the lattice of all normal Fitting classes into the lattice of the Lockett section generated by $\mathfrak{X}_*\mathfrak{Y}$. The proof of the lemma is complete.

§ 2. p-Local $\mathscr{L}$-Classes

In the next theorem we prove the surjectivity of the mapping from the lattice of all normal Fitting classes into the lattice of the Lockett section generated by the p-local Fitting classes. This confirms the existence of p-local $\mathscr{L}$-classes which are not Lockett classes.

Theorem 1. *Take $\mathfrak{Y} = (\mathfrak{S}_{p'})_*\mathfrak{N}_p$. Then $\mathfrak{Y}$ is a p-local Fitting class but not a Lockett class, and the mapping is surjective from the lattice of all normal Fitting classes into* Locksec($\mathfrak{Y}$).

Proof. The Fitting class $\mathfrak{Y}$ is p-local. Indeed, it is clear that $\mathfrak{Y}(F^p) \subseteq (\mathfrak{S}_{p'})_*$, and consequently $\mathfrak{Y}(F^p)\mathfrak{N}_p \subseteq (\mathfrak{S}_{p'})_*\mathfrak{N}_p = \mathfrak{Y}$. By Lemma 6 the latter means that the Fitting class $\mathfrak{Y}$ is p-local.

In order to prove the surjectivity of the mapping from the lattice of all normal Fitting classes into Locksec($\mathfrak{Y}$), by [6, X, 1.19; X, 6.1] it suffices to prove that the Fitting class $\mathfrak{Y}$ is an $\mathscr{L}$-class.

Let us show that $\mathfrak{Y}$ is an $\mathscr{L}$-class. Since the mapping is surjective from the lattice of all normal Fitting classes into Locksec($\mathfrak{S}_{p'}\mathfrak{N}_p$), and $\mathfrak{S}_{p'}$ is a Lockett class, the mapping is surjective from the lattice of all normal Fitting classes into Locksec($(\mathfrak{S}_{p'})^*\mathfrak{N}_p$). Consequently, since the mapping from the lattice of all normal Fitting classes into Locksec($\mathfrak{S}_{p'}$) is surjective and $\mathfrak{N}_p$ is a saturated radical formation, it follows by Lemma 8 that the mapping from the lattice of all normal Fitting classes into Locksec($\mathfrak{Y}$) is surjective.

In order to verify that $\mathfrak{Y}$ is not a Lockett class, assume on the contrary that the Fitting class $\mathfrak{Y}$ is a Lockett class: $((\mathfrak{S}_{p'})_*\mathfrak{N}_p)^* = (\mathfrak{S}_{p'})_*\mathfrak{N}_p$. Since $\mathfrak{N}_p$ is a saturated radical homomorph,

$$((\mathfrak{S}_{p'})_*\mathfrak{N}_p)^* = ((\mathfrak{S}_{p'})_*)^*\mathfrak{N}_p = (\mathfrak{S}_{p'})^*\mathfrak{N}_p = \mathfrak{S}_{p'}\mathfrak{N}_p.$$

We deduce that $\mathfrak{S}_{p'}\mathfrak{N}_p = (\mathfrak{S}_{p'})_*\mathfrak{N}_p$.

Since $\mathfrak{S}_{p'}$ satisfies Lockett's conjecture, $(\mathfrak{S}_{p'})_* = \mathfrak{S}_{p'} \cap \mathfrak{S}_*$. Consequently, $(\mathfrak{S}_{p'} \cap \mathfrak{S}_*)\mathfrak{N}_p = \mathfrak{S}_{p'}\mathfrak{N}_p$. Since $\mathfrak{N}_p$ is a saturated radical formation, $(\mathfrak{S}_{p'} \cap \mathfrak{S}_*)\mathfrak{N}_p = \mathfrak{S}_{p'}\mathfrak{N}_p \cap \mathfrak{S}_*\mathfrak{N}_p$. Hence, $\mathfrak{S}_{p'}\mathfrak{N}_p \cap \mathfrak{S}_*\mathfrak{N}_p = \mathfrak{S}_{p'}\mathfrak{N}_p$. Therefore, $\mathfrak{S}_{p'}\mathfrak{N}_p \subseteq \mathfrak{S}_*\mathfrak{N}_p$.

It is obvious that $\mathfrak{S}_{p'} \subseteq \mathfrak{S}_*\mathfrak{N}_p$. This implies the inclusion $\mathfrak{S}_{p'} \cap \mathfrak{S}_*\mathfrak{S}_{p'} \subseteq \mathfrak{S}_*\mathfrak{N}_p \cap \mathfrak{S}_*\mathfrak{S}_{p'}$. Clearly, $\mathfrak{S}_{p'} \cap \mathfrak{S}_*\mathfrak{S}_{p'} = \mathfrak{S}_{p'}$, while $\mathfrak{S}_*\mathfrak{N}_p \cap \mathfrak{S}_*\mathfrak{S}_{p'} = \mathfrak{S}_*(\mathfrak{N}_p \cap \mathfrak{S}_{p'}) = \mathfrak{S}_*$. Consequently, $\mathfrak{S}_{p'} \subseteq \mathfrak{S}_*$. Lemma 5 yields a contradiction.

Thus, the Fitting class $\mathfrak{Y}$ is not a Lockett class. The proof of the theorem is complete.

§ 3. $\overline{\mathscr{L}}$-Classes

Observe that in the general case the mapping is not surjective from the lattice of all normal Fitting classes into the lattice of the Lockett section generated by partially local Fitting classes. To construct an example of a mapping of this type we use the Berger–Cossey class $\mathfrak{B}$ [9]. Recall the main stages of its construction.

Take the extra special group R of order 27 and exponent 3, and a faithful irreducible 3-dimensional R-module over the field $GF(7)$. Suppose that $Y = WR$. Denote the automorphism group of R by A. Put $B = C_A(Z(R))$, denote the quaternion subgroup of B by Q, and take $X = Z(Q)Y$.

Following [9], define the class $\mathfrak{M} = (G \mid O_2(G/O_{\{2,3\}}(G)) \in S_nD_0(X))$, where $D_0(X)$ is the class of all finite direct products of isomorphic copies of X, and $\mathfrak{B} = \mathfrak{M} \cap \mathfrak{S}_7\mathfrak{S}_3\mathfrak{S}_2$. It is established in [9] that $\mathfrak{B}$ is a Lockett class and $\mathfrak{B}_* \neq \mathfrak{B} \cap \mathfrak{S}_*$. By [6, X, 6.1] this means that the mapping is not surjective from the lattice of all normal Fitting classes into Locksec($\mathfrak{F}$).

Theorem 2. *There exists a prime p such that the mapping is not surjective from the lattice of all normal Fitting classes into the lattice of the Lockett section generated by the p-local Fitting class $\mathfrak{B}_*\mathfrak{N}_p$.*

Proof. Suppose that for every prime p the mapping is surjective from the lattice of all normal Fitting classes into $\mathrm{Locksec}(\mathfrak{B}_*\mathfrak{N}_p)$. By [6, X, 6.1] this is equivalent to the validity of $(\mathfrak{B}_*\mathfrak{N}_p)_* = (\mathfrak{B}_*\mathfrak{N}_p)^* \cap \mathfrak{S}_*$ for all p. Since $\mathfrak{N}_p$ is a saturated radical homomorph, Lemmas 1 and 2 imply that

$$\bigcap_{p\in\mathbb{P}}(\mathfrak{B}_*\mathfrak{N}_p)_* = \bigcap_{p\in\mathbb{P}}(\mathfrak{B}^*\mathfrak{N}_p \cap \mathfrak{S}_*).$$

However,

$$\bigcap_{p\in\mathbb{P}}(\mathfrak{B}^*\mathfrak{N}_p \cap \mathfrak{S}_*) = \bigcap_{p\in\mathbb{P}}(\mathfrak{B}^*\mathfrak{N}_p) \cap \mathfrak{S}_* = \mathfrak{B}^* \cap \mathfrak{S}_*.$$

Verify now that $\bigcap_{p\in\mathbb{P}}(\mathfrak{B}_*\mathfrak{N}_p)_* = \mathfrak{B}_*$. Lemma 2(2) implies that $(\mathfrak{B}_*\mathfrak{N}_p)_* \subseteq \mathfrak{B}_*\mathfrak{N}_p$ for every $p \in \mathbb{P}$. Consequently,

$$\bigcap_{p\in\mathbb{P}}(\mathfrak{B}_*\mathfrak{N}_p)_* \subseteq \bigcap_{p\in\mathbb{P}}(\mathfrak{B}_*\mathfrak{N}_p) = \mathfrak{B}_*.$$

On the other hand, $\mathfrak{B}_* \subseteq \mathfrak{B}_*\mathfrak{N}_p$ for every $p \in \mathbb{P}$. However, by Lemma 2(1) $(\mathfrak{B}_*)_* = \mathfrak{B}_* \subseteq (\mathfrak{B}_*\mathfrak{N}_p)_*$ for every $p \in \mathbb{P}$. Consequently, $\mathfrak{B}_* \subseteq \bigcap_{p\in\mathbb{P}}(\mathfrak{B}_*\mathfrak{N}_p)_*$. Hence,

$$\mathfrak{B}_* = \bigcap_{p\in\mathbb{P}}(\mathfrak{B}_*\mathfrak{N}_p)_*.$$

Therefore, $\mathfrak{B}_* = \mathfrak{B}^* \cap \mathfrak{S}_*$. This contradicts the fact that the Fitting class $\mathfrak{B}$ is an $\overline{\mathscr{L}}$-class. Thus, there is a prime p such that $\mathfrak{B}_*\mathfrak{N}_p$ is an $\overline{\mathscr{L}}$-class. By [6, X, 1.19] this means that the mapping is not surjective from the lattice of all normal Fitting classes into $\mathrm{Locksec}(\mathfrak{B}_*\mathfrak{N}_p)$. The proof of the theorem is complete.

§ 4. ω-Local $\mathscr{L}_{\mathfrak{H}}$-Classes

The next theorem gives a sufficient condition for the surjectivity of the mapping from the lattice of the Lockett section generated by arbitrary Fitting classes into the lattice of the Lockett section generated by ω-local Fitting classes. We verify this result in the class of all finite groups.

Theorem 3. *Take an ω-local Fitting class $\mathfrak{F}$ with $\mathfrak{F} \subseteq \mathfrak{X}$, where $\mathfrak{X}$ is a Fitting class. If* $\mathrm{Char}(\mathfrak{F}) \subseteq \omega$ *then the mapping is surjective from* $\mathrm{Locksec}(\mathfrak{X})$ *into* $\mathrm{Locksec}(\mathfrak{F})$.

Proof. By [6, X, 1.19; X, 6.1] to prove the surjectivity of the mapping from $\mathrm{Locksec}(\mathfrak{X})$ into $\mathrm{Locksec}(\mathfrak{F})$, it suffices to prove that $\mathfrak{F}$ is an $\mathscr{L}_{\mathfrak{X}}$-class.

Let us show that $\mathfrak{F}$ is an $\mathscr{L}_{\mathfrak{X}}$-class. By a result of Bryce and Cossey [4], a necessary and sufficient condition for that is the validity of (2): $\mathfrak{F}_* = \mathfrak{F}^* \cap \mathfrak{X}_*$. Since $\mathfrak{F}$ is an ω-local class, Lemma 6 implies that $\mathfrak{F}(F^p)\mathfrak{N}_p \subseteq \mathfrak{F}$ for all $p \in \omega$, and so for every $p \in \mathrm{Char}(\mathfrak{F})$ as well.

By the ω-locality $\mathfrak{F}$ is defined using an ω-local H-function f as

$$\mathfrak{F} = \Big(\bigcap_{p\in\pi_2}\mathfrak{E}_{p'}\Big) \cap \Big(\bigcap_{p\in\pi_1} f(p)\mathfrak{N}_p\mathfrak{E}_{p'}\Big) \cap f(\omega')\mathfrak{E}_\omega,$$

where $\pi_1 = \omega \cap \mathrm{Supp}(f)$ and $\pi_2 = \omega \setminus \pi_1$. Consequently, $\mathfrak{F} \subseteq f(p)\mathfrak{N}_p\mathfrak{E}_{p'}$ for all $p \in \pi_1$. However, $f(p) = \mathfrak{F}(F^p)\mathfrak{N}_p$ for all $p \in \omega$ by Lemma 6, and thereby $\mathfrak{F} \subseteq \mathfrak{F}(F^p)\mathfrak{N}_p\mathfrak{E}_{p'}$ for all $p \in \mathrm{Supp}(f) \cap \omega$. In this case

$$\mathfrak{F}(F^p) = \begin{cases} \mathrm{Fit}(F^p(G) \mid G \in \mathfrak{F}), & p \in \pi(\mathfrak{F}),\\ \varnothing, & p \notin \pi(\mathfrak{F}).\end{cases}$$

Since $\mathrm{Char}(\mathfrak{F}) \subseteq \omega$,

$$\mathfrak{F}(F^p)\mathfrak{N}_p \subseteq \mathfrak{F} \subseteq \mathfrak{F}(F^p)\mathfrak{N}_p\mathfrak{E}_{p'}$$

for every $p \in \mathrm{Char}(\mathfrak{F})$. Consequently, $\mathfrak{F}$ is a Lockett class by Lemma 4. Hence, by [13, Theorem 1] $\mathfrak{F}$ is defined by the largest reduced ω-local H-function F, and moreover $F(p)\mathfrak{N}_p = F(p) \subseteq \mathfrak{F}$ for all $p \in \omega$.

Arguing similarly, we conclude that $F(p)\mathfrak{N}_p \subseteq \mathfrak{F} \subseteq F(p)\mathfrak{N}_p\mathfrak{E}_{p'}$ for all $p \in \mathrm{Char}(\mathfrak{F})$.

Verify now that $\mathfrak{F} \subseteq \mathfrak{F}_*\mathfrak{E}_{p'}\mathfrak{N}_p$. Since $\mathfrak{E}_{p'}\mathfrak{N}_p$ is a saturated Fitting formation, by Lemma 1 we have $(\mathfrak{F}_*\mathfrak{E}_{p'}\mathfrak{N}_p)^* = (\mathfrak{F}_*)^*\mathfrak{E}_{p'}\mathfrak{N}_p$. However, $(\mathfrak{F}_*)^* = \mathfrak{F}^*$ by Lemma 2(2). Consequently, $(\mathfrak{F}_*\mathfrak{E}_{p'}\mathfrak{N}_p)^* = \mathfrak{F}^*\mathfrak{E}_{p'}\mathfrak{N}_p$. Since $\mathfrak{F}_*\mathfrak{E}_{p'}\mathfrak{N}_p$ and $\mathfrak{F}\mathfrak{E}_{p'}\mathfrak{N}_p$ are local classes (see [8, Corollary 1]), they are Lockett classes by [8, Lemma 5]. Consequently, $\mathfrak{F}_*\mathfrak{E}_{p'}\mathfrak{N}_p = \mathfrak{F}\mathfrak{E}_{p'}\mathfrak{N}_p$, and so $\mathfrak{F} \subseteq \mathfrak{F}_*\mathfrak{E}_{p'}\mathfrak{N}_p$.

This implies that $G/G_{\mathfrak{F}_*\mathfrak{E}_{p'}} \in \mathfrak{N}_p$ for all groups $G \in \mathfrak{F}$. Moreover, the inclusion $\mathfrak{F} \subseteq F(p)\mathfrak{N}_p\mathfrak{E}_{p'}$ implies that $G/G_{F(p)\mathfrak{N}_p} \in \mathfrak{E}_{p'}$ for all $G \in \mathfrak{F}$.

Verify now that $\mathfrak{X}_* \cap \mathfrak{F} \subseteq \mathfrak{F}_*\mathfrak{E}_{p'}$ for all $p \in \mathrm{Char}(\mathfrak{F})$. To this end, establish firstly that

$$(\mathfrak{F}_*\mathfrak{E}_{p'} \cap \mathfrak{F}) \vee F(p)\mathfrak{N}_p = \mathfrak{F}.$$

The inclusion $(\mathfrak{F}_*\mathfrak{E}_{p'} \cap \mathfrak{F}) \vee F(p)\mathfrak{N}_p \subseteq \mathfrak{F}$ is obvious. In order to verify the reverse inclusion, take $G \in \mathfrak{F}$. Then $G/G_{\mathfrak{F}_*\mathfrak{E}_{p'}} \in \mathfrak{N}_p$ and $G/G_{F(p)\mathfrak{N}_p} \in \mathfrak{E}_{p'}$. This implies that $G/G_{F(p)\mathfrak{N}_p}G_{\mathfrak{F}_*\mathfrak{E}_{p'}} \in \mathfrak{N}_p \cap \mathfrak{E}_{p'} = (1)$. Hence, $G = G_{F(p)\mathfrak{N}_p}G_{\mathfrak{F}_*\mathfrak{E}_{p'}}$. However, $G_{\mathfrak{F}_*\mathfrak{E}_{p'}} = G_{\mathfrak{F}_*\mathfrak{E}_{p'}} \cap G = G_{\mathfrak{F}_*\mathfrak{E}_{p'}} \cap G_{\mathfrak{F}} = G_{\mathfrak{F}_*\mathfrak{E}_{p'} \cap \mathfrak{F}}$. Thus, if $G \in \mathfrak{F}$ then $G = G_{F(p)\mathfrak{N}_p}G_{\mathfrak{F}_*\mathfrak{E}_{p'} \cap \mathfrak{F}}$. Hence, $G \in (\mathfrak{F}_*\mathfrak{E}_{p'} \cap \mathfrak{F}) \vee F(p)\mathfrak{N}_p$. Therefore, we have established that

$$\mathfrak{F} = (\mathfrak{F}_*\mathfrak{E}_{p'} \cap \mathfrak{F}) \vee F(p)\mathfrak{N}_p.$$

Thus, by Lemma 3 $\mathfrak{X}_* \cap \mathfrak{F} \subseteq \mathfrak{F}_*\mathfrak{E}_{p'}$ for all $p \in \mathrm{Char}(\mathfrak{F})$.

It remains to verify that if $\mathfrak{X}_* \cap \mathfrak{F} \subseteq \mathfrak{F}_*\mathfrak{E}_{p'}$ for all $p \in \mathrm{Char}(\mathfrak{F})$ then $\mathfrak{X}_* \cap \mathfrak{F} = \mathfrak{F}_*$. It is obvious that $\mathfrak{F}_* \subseteq \mathfrak{X}_* \cap \mathfrak{F}$. Take a group G of the smallest order in the class $(\mathfrak{X}_* \cap \mathfrak{F}) \setminus \mathfrak{F}_*$. Then G has a unique maximal normal subgroup $M = G_{\mathfrak{F}_*}$. Consider the quotient G/M and take a prime divisor p of $|G/M|$.

Since $G \in \mathfrak{F}$, Lemma 2(3) implies that $G/G_{\mathfrak{F}_*}$ is an abelian group. Consequently, G/M is a composition factor of order p; i.e., $G/M \simeq Z_p \in \mathfrak{N}_p$. Hence, $p \in \mathrm{Char}(\mathfrak{F})$. However, $G \in \mathfrak{F}_*\mathfrak{E}_{p'}$ by the argument above; hence, $G/M \in \mathfrak{E}_{p'}$. Thus, $G/M \in \mathfrak{N}_p \cap \mathfrak{E}_{p'} = (1)$ and $G = M \in \mathfrak{F}_*$, which contradicts the assumption that $G \notin \mathfrak{F}_*$. Therefore, $\mathfrak{X}_* \cap \mathfrak{F} \subseteq \mathfrak{F}_*$. Consequently, $\mathfrak{X}_* \cap \mathfrak{F} = \mathfrak{F}_*$. Taking Lemma 4 into account, we have $\mathfrak{F}^* = \mathfrak{F}$. Therefore, $\mathfrak{F}$ is an $\mathscr{L}_{\mathfrak{X}}$-class, and the mapping is surjective from $\mathrm{Locksec}(\mathfrak{X})$ into $\mathrm{Locksec}(\mathfrak{F})$. The proof of the theorem is complete.

Observe that if ω coincides with the set $\mathbb{P}$ of all primes then every ω-local Fitting class is local. However, not every ω-local Fitting class is local. (For instance, the Fitting class $\mathfrak{X} = \mathfrak{F}\mathfrak{N}_p$, where $\mathfrak{F}$ is an arbitrary nontrivial normal Fitting class, is an ω-local but not local Fitting class for $\omega = \{p\}$.) It is easy to see that every solvable ω-local Fitting class $\mathfrak{F}$ with $\mathrm{Char}(\mathfrak{F}) \subseteq \omega$ is local. Moreover, by [8, Lemma 5] every local Fitting class is a Lockett class. This raises the question of the existence in the class $\mathfrak{E}$ of all finite groups of ω-local Fitting classes $\mathfrak{F}$ with $\mathrm{Char}(\mathfrak{F}) \subseteq \omega$ which are nonlocal. The positive answer to this question is given by

Example 1. Take a nonabelian simple group E, the Fitting class $\mathfrak{X} = \mathrm{Fit}\, E$ generated by E, and a prime p. Put $\mathfrak{F} = \mathfrak{X}\mathfrak{N}_p$ and $\omega = \{p\}$. Then $\mathfrak{F}$ is an ω-local Lockett class with $\mathrm{Char}(\mathfrak{F}) \subseteq \omega$ which is nonnormal and nonlocal.

Indeed, since $\mathfrak{F}(F^p) \subseteq \mathfrak{X}$, and consequently $\mathfrak{F}(F^p)\mathfrak{N}_p \subseteq \mathfrak{X}\mathfrak{N}_p = \mathfrak{F}$, Lemma 6 implies that $\mathfrak{F}$ is an ω-local Fitting class for $\omega = \{p\}$.

Since $\mathfrak{X}$ consists only of the trivial group and the finite direct products of groups isomorphic to E, it follows that $\mathfrak{X}$ is a Fitting formation. Hence, $\mathfrak{F} = \mathfrak{X}\mathfrak{N}_p$ is a Fitting formation, and every group in $\mathfrak{F}$ is either a p-group (possibly the trivial one) or an extension of a finite direct product of groups isomorphic to E by a p-group (possibly the trivial one). This implies that $\mathrm{Char}(\mathfrak{F}) = \{p\}$, while by [6, X, 1.25] $\mathfrak{F}$ is the Lockett class.

Verify now that $\mathfrak{F}$ is nonlocal. Suppose that $\mathfrak{F} = LR(f)$, where f is a complete reduced H-function. Then

$$\mathfrak{F} = \mathfrak{E}_\pi \cap \Big(\bigcap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{E}_{p'}\Big),$$

where $\pi = \mathrm{Supp}(f)$. By [10, 4.9b)] we have $\mathrm{Char}(\mathfrak{F}) = \pi(\mathfrak{F})$. Since in this case $\mathrm{Char}(\mathfrak{F}) = \{p\}$, while $|\pi(\mathfrak{F})| \geq 2$, we obtain a contradiction with $\mathrm{Char}(\mathfrak{F}) = \pi(\mathfrak{F})$. Consequently, $\mathfrak{F}$ is a nonlocal class.

Verify that $\mathfrak{F}$ is not normal. Indeed, if $\mathfrak{F}$ is a normal Fitting class then $\mathrm{Char}(\mathfrak{F}) = \mathbb{P}$ by [6, X, 3.2]. This contradicts $\mathrm{Char}(\mathfrak{F}) = \{p\}$.

Consequently, if $\omega = \{p\}$, where $p \in \mathbb{P}$, then $\mathfrak{F}$ is an ω-local Lockett class with $\mathrm{Char}(\mathfrak{F}) \subseteq \omega$ which is nonnormal and nonlocal.

Therefore, in the case $\omega = \mathbb{P}$ Theorem 3 implies the result of Gallego [10], which we include as a corollary.

Corollary 1 [10]. *Each local Fitting class is an $\mathscr{L}_{\mathfrak{E}}$-class.*

In the case $\mathfrak{X} = \mathfrak{S}$, Theorem 3 yields

Corollary 2. *Take two ω-local Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ with $\mathrm{Char}(\mathfrak{F}) \subseteq \omega$ and $\mathrm{Char}(\mathfrak{H}) \subseteq \omega$. Then $(\mathfrak{F} \cap \mathfrak{H})_* = (\mathfrak{F} \cap \mathfrak{H}) \cap \mathfrak{S}_*$.*

Proof. Since $\mathrm{Char}(\mathfrak{F}) \subseteq \omega$, Theorem 3 implies that $\mathfrak{F}$ is an $\mathscr{L}$-class. Similarly, $\mathfrak{H}$ is an $\mathscr{L}$-class.

By [11, Lemma 21] the intersection $\mathfrak{F} \cap \mathfrak{H}$ of ω-local Fitting classes is an ω-local Fitting class, while $\mathrm{Char}(\mathfrak{F} \cap \mathfrak{H}) \subseteq \omega$. Consequently, by Theorem 3 the ω-local Fitting class $\mathfrak{F} \cap \mathfrak{H}$ is an $\mathscr{L}$-class; i.e., $(\mathfrak{F} \cap \mathfrak{H})_* = (\mathfrak{F} \cap \mathfrak{H})^* \cap \mathfrak{S}_*$.

Since by Theorem 3 the Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ are Lockett classes, Lemma 2(4) implies that $\mathfrak{F} \cap \mathfrak{H}$ is a Lockett class. Therefore, $(\mathfrak{F} \cap \mathfrak{H})_* = (\mathfrak{F} \cap \mathfrak{H}) \cap \mathfrak{S}_*$. The proof of the corollary is complete.

Observe that Corollary 2 yields a positive answer to a question of Lausch (see [2, Problem 8.30]) in the case of ω-local Fitting classes whose characteristic is a subset of ω.

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