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On the Distributivity of the Lattice of Solvable Totally Local Fitting Classes

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ABSTRACT. It is proved that the lattice of all solvable totally local Fitting classes is algebraic and distributive.

KEY WORDS: formation, formation function, satellite, Fitting class, solvable class, totally local class, algebraic lattice, distributive lattice.

All groups under consideration are finite. Recall that functions of the form $f\colon \mathbb{P} \mapsto \{\text{formations}\}$ are called *formation functions* [1] or *satellites* [2]. If a class of groups $\mathfrak{F}$ is of the form

$$\mathfrak{F} = \mathfrak{G}_{\pi(\mathfrak{F})} \cap \Big(\bigcap_{p\in\pi(\mathfrak{F})} \mathfrak{G}_{p'}\mathfrak{N}_p f(p)\Big),$$

where f is a satellite, then we say that $\mathfrak{F}$ is a local formation with satellite f and write $\mathfrak{F} = \mathrm{LF}(f)$. Here $\pi(\mathfrak{F})$ is the set of all prime divisors of the orders of the groups in $\mathfrak{F}$, and the symbols $\mathfrak{N}_p$ and $\mathfrak{G}_{p'}$ stand for the class of all p-groups and the class of all p'-groups, respectively.

Functions of the form $f\colon \mathbb{P} \mapsto \{\text{Fitting classes}\}$ are called *Hartley functions* or briefly *H-functions* [2]. If for a class of groups $\mathfrak{F}$ we have

$$\mathfrak{F} = \mathfrak{G}_{\pi(\mathfrak{F})} \cap \Big(\bigcap_{p\in\pi(\mathfrak{F})} f(p)\mathfrak{N}_p\mathfrak{G}_{p'}\Big),$$

where f is an H-function, then we say that $\mathfrak{F}$ is a *local Fitting class* with H-function f and write $\mathfrak{F} = \mathrm{LR}(f)$.

When surveying the best-known specific classes of groups, it can readily be discovered that these classes can be defined by means of functions whose nonempty values are some local classes themselves. This fact led to the following natural construction [3]: each formation is assumed to be 0-fold local and, for $n \geq 1$, a formation $\mathfrak{F}$ is said to be *n-fold local* if $\mathfrak{F} = \mathrm{LF}(f)$, where all the nonempty values of the satellite f are $(n-1)$-fold local. A formation is said to be *totally local* if it is n-fold local for all positive integers n. The n-fold local and totally local Fitting classes are defined in a similar way. We can readily show that the class of solvable totally local formations coincides with the class of so-called primitive saturated formations introduced by Hawkes in [4]. Multiply local classes have applications in the solution of many problems in the theory of classes (see, e.g., [5–10]).

The totally local classes are the limit case of n-fold local classes and have specific properties. In particular, we note that, for any nonnegative integer n, the lattices of all n-fold local formations, of the n-fold local hereditary formations, of the n-fold local normally hereditary formations, etc., are modular, but all these formations are not distributive even in the class of solvable groups $\mathfrak{S}$ (see [5, Chap. 2] and [6, Chap. 4]). As far as the totally local formations are concerned, it is not known at present whether the lattice of all totally local τ-closed formations is at least modular for at least one nontrivial subgroup functor τ [6, Question 4.2.14]. At the same time, in [5] it was announced that the lattice of all solvable totally local formations is distributive. In the present paper we prove that the lattice of all solvable totally local Fitting classes is algebraic and distributive, and each of its elements that differs from (1) and $\mathfrak{S}$ is not complementable. In the course of proving this result, we clarify some general properties of the generation operator $\vee^{\infty}$. On the other hand, as one of the consequences, we give here a complete proof

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of the distributivity of the lattice of all solvable totally local formations; another scheme of such a proof was discussed in the monograph [6].

Let us recall some definitions [2] related to local Fitting classes. By the symbol $F^p(G)$ we denote the subgroup $O^p(O^{p'}(G))$. By l^∞ we denote the lattice of all totally local Fitting classes. An H-function f is said to be *l^∞-valued* [6] if each nonempty value of f belongs to the lattice l^∞. Let $\{f_i(p) \mid i \in I\}$ be an arbitrary collection of l^∞-valued H-functions. For any $p \in \mathbb{P}$ we set

$$\Big(\bigcap_{i\in I} f_i\Big)(p) = \bigcap_{i\in I} f_i(p).$$

An H-function $\bigcap_{i\in I} f_i$ is called the *intersection* of the H-functions f_i. If a Fitting class $\mathfrak{F}$ admits at least one l^∞-valued H-function, then the intersection of all such H-functions for the class $\mathfrak{F}$ is called the *minimal l^∞-valued H-function* of the class $\mathfrak{F}$.

Let $\mathfrak{X}$ be an arbitrary nonempty family of groups. The intersection of all totally local Fitting classes containing $\mathfrak{X}$ is denoted by $l^\infty \operatorname{fit} \mathfrak{X}$ and is called the *totally local Fitting class* generated by $\mathfrak{X}$ [2]. If $\mathfrak{X} = \{G\}$, then we write $l^\infty \operatorname{fit} G$ instead of $l^\infty \operatorname{fit}\{G\}$. Each Fitting class of this form is called a *singly generated totally local Fitting class* [2].

We need the following special case of Lemma 21 in [2].

Lemma 1. *If $\mathfrak{F} = l^\infty \operatorname{fit} \mathfrak{X}$ and if f is the minimal l^∞-valued H-function for the class $\mathfrak{F}$, then*

$$f(p) = l^\infty \operatorname{fit}(F^p(G) \mid G \in \mathfrak{X})$$

for all $p \in \pi(\mathfrak{X})$, and $f(p) = \varnothing$ for any $p \in \mathbb{P} \setminus \pi(\mathfrak{X})$.

Proof. Let t be an H-function such that $t(p) = l^\infty \operatorname{fit}(F^p(G) \mid G \in \mathfrak{X})$ for any $p \in \pi(\mathfrak{X})$ and $t(p) = \varnothing$ for any $p \in \mathbb{P} \setminus \pi(\mathfrak{X})$. Let us show that $t = f$. Let $\mathfrak{M} = \mathrm{LR}(t)$. Then it is clear that $\mathfrak{X} \subseteq \mathfrak{M}$. Hence, $\mathfrak{F} \subseteq \mathfrak{M}$. Let f_1 be an arbitrary l^∞-valued H-function of the class $\mathfrak{F}$. Then, since $\mathfrak{X} \subseteq \mathfrak{F}$, it follows that $(F^p(G) \mid G \in \mathfrak{X}) \subseteq f_1(p)$ for any $p \in \mathbb{P}$. Therefore, $t(p) \subseteq f_1(p)$. Consequently, $\mathfrak{M} \subseteq \mathfrak{F}$. Hence, $\mathfrak{M} = \mathfrak{F}$ and $t = f$. This proves the lemma. □

Let $\{\mathfrak{F}_i \mid i \in I\}$ be an arbitrary family of Fitting classes in l^∞. In this case, by $\vee^\infty(\mathfrak{F}_i \mid i \in I)$ we denote [6] the least upper bound of $\{\mathfrak{F}_i \mid i \in I\}$ in l^∞ and by $\vee^\infty(f_i \mid i \in I)$ the H-function f such that $f(p)$ is the least upper bound of $\{f_i(p) \mid i \in I\}$ in l^∞ if $\bigcup_{i\in I} f_i(p) \neq \varnothing$, and $f(p) = \varnothing$ otherwise.

Lemma 2. *Let f_i be the minimal l^∞-valued H-function for the Fitting class $\mathfrak{F}_i$, $i \in I$. Then the function $\vee^\infty(f_i \mid i \in I)$ is the minimal l^∞-valued H-function for the Fitting class $\mathfrak{F} = \vee^\infty(\mathfrak{F}_i \mid i \in I)$.*

Proof. Let

$$\pi = \pi\Big(\bigcup_{i\in I} \mathfrak{F}_i\Big) = \bigcup_{i\in I} \pi(\mathfrak{F}_i) = \pi(\mathfrak{F}),$$

let $f = \vee^\infty(f_i \mid i \in I)$, and let h be the minimal l^∞-valued H-function for the Fitting class $\mathfrak{F}$. Let us show that $h = f$.

Let $p \in \mathbb{P} \setminus \pi$. In this case, for any $i \in I$, we have $h(p) = \varnothing$ and $f_i(p) = \varnothing$. Hence, $f(p) = \varnothing$.

Let $p \in \pi$. In this case, there is an $i \in I$ such that $f_i(p) \neq \varnothing$. By Lemma 1 we have

$$h(p) = l^\infty \operatorname{fit}\Big(\Big(F^p(G) \mid G \in \bigcup_{i\in I} \mathfrak{F}_i\Big)\Big) = l^\infty \operatorname{fit}\Big(\Big(\bigcup_{i\in I} l^\infty \operatorname{fit}((F^p(G) \mid G \in \mathfrak{F}_i))\Big)\Big)$$
$$= l^\infty \operatorname{fit}\Big(\Big(\bigcup_{i\in I} f_i(p)\Big)\Big) = \big((\vee^\infty(f_i \mid i \in I))\big)(p) = f(p).$$

This proves the lemma. □

An H-function f is said to be *inner* if $f(p) \subseteq \mathrm{LR}(f)$ for any $p \in \mathbb{P}$. In what follows, we need the following lemma, which is a special case of the result obtained by Vorob′ev in [11].

Lemma 3. *For any totally local Fitting class* $\mathfrak{H}$, *the class* $\mathfrak{H}\mathfrak{N}_p$ *totally local.*

Lemma 4 [2, Lemma 23]. *Let* $\mathfrak{F} = \mathrm{LR}(f)$. *In this case, if* $O^p(G) \in f(p) \cap \mathfrak{F}$, *then* $G \in \mathfrak{F}$.

To any Fitting class we can assign the smallest (by inclusion) Fitting class $\mathfrak{F}^*$ [12] that contains $\mathfrak{F}$ and satisfies $(G \times H)_{\mathfrak{F}^*} = G_{\mathfrak{F}^*} \times H_{\mathfrak{F}^*}$ for any groups G and H. A Fitting class $\mathfrak{F}$ is called a *Lockett class* if $\mathfrak{F} = \mathfrak{F}^*$.

Lemma 5. *If* $\{\mathfrak{F}_i = \mathrm{LR}(f_i) \mid i \in I\}$ *is a set of totally local Fitting classes, where* f_i *is an inner* l^∞-*valued* H-*function, then*

$$\vee^\infty(\mathfrak{F}_i \mid i \in I) = \mathrm{LR}(\vee^\infty(f_i \mid i \in I)).$$

Proof. Let $\mathfrak{F} = \vee^\infty(\mathfrak{F}_i \mid i \in I)$, let $\mathfrak{M} = \mathrm{LR}(\vee^\infty(f_i \mid i \in I))$, and let h_i be the minimal l^∞-valued H-function for the class $\mathfrak{F}_i$. Then, by Lemma 2, $h = \vee^\infty(h_i \mid i \in I)$ is the minimal l^∞-valued H-function for the class $\mathfrak{F}$. Since $h_i \le f_i$, it follows that, for any $p \in \mathbb{P}$, the following inclusion holds:

$$l^\infty \operatorname{fit}\Big(\bigcup_{i\in I} h_i(p)\Big) \subseteq l^\infty \operatorname{fit}\Big(\bigcup_{i\in I} f_i(p)\Big).$$

Hence, $\mathfrak{F} \subseteq \mathfrak{M}$.

Now let us prove the converse inclusion. Let t_i be an H-function for $\mathfrak{F}_i$ such that $t_i(p) = h_i(p)\mathfrak{N}_p$ for any $p \in \mathbb{P}$. By Lemma 3, this H-function is l^∞-valued. Let us show that $f_i \le t_i$.

Assume that $f_i \not\le t_i$. Then there is a prime number p such that $f_i(p) \not\subseteq t_i(p)$. Let G be a group in $f_i(p) \setminus t_i(p)$. Let $\Gamma = G \wr Z_p = [K]Z_p$, where Z_p is a group of order p and K is the base of the regular wreath product Γ. Since each local Fitting class is a Lockett class [13] and $G \notin t_i(p)$, it follows from [1, Chap. X, Proposition 2.1 a)] that $\Gamma_{t_i(p)} = K_1$, where K_1 is the base of the regular wreath product $\Gamma_1 = G_{t_i(p)} \wr Z_p$. It follows from the properties of the wreath products (see, e.g., [1, Chap. A, 18.2 d)]) that

$$\Gamma/\Gamma_{t_i(p)} = \Gamma/K_1 \simeq (G/G_{t_i(p)}) \wr Z_p.$$

Hence, the order of the group $\Gamma/\Gamma_{t_i(p)}$ is divisible by p.

Since $G \in f_i(p)$, it follows that $K \in f_i(p)$. Therefore, $K \subseteq \Gamma_{f_i(p)}$. Since $\Gamma/K \simeq Z_p \in \mathfrak{N}_p$, it follows that

$$\Gamma/K/\Gamma_{f_i(p)}/K \simeq \Gamma/\Gamma_{f_i(p)} \in \mathfrak{N}_p.$$

Hence, by Lemma 4 we obtain $\Gamma \in f_i(p)\mathfrak{N}_p \subseteq \mathfrak{F}_i = \mathrm{LR}(f_i) = \mathrm{LR}(t_i)$, and therefore

$$\Gamma \in \mathfrak{G}_{\pi(\mathfrak{F})} \cap \Big(\bigcap_{q\in\pi(\mathfrak{F})} t_i(q)\mathfrak{G}_{q'}\Big)$$

and, in particular, $\Gamma \in t_i(p)\mathfrak{G}_{p'}$ for any $p \in \pi(\mathfrak{F})$. Consequently, $\Gamma/\Gamma_{t_i(p)} \in \mathfrak{G}_{p'}$. A contradiction.

Thus, $f_i \le t_i$. Hence, $f = \vee^\infty(f_i \mid i \in I) \le \vee^\infty(t_i \mid i \in I)$, i.e., for any $p \in \mathbb{P}$ we have the inclusion

$$f(p) = \vee^\infty(f_i(p) \mid i \in I) \subseteq \vee^\infty(t_i(p) \mid i \in I) = \vee^\infty(h_i(p)\mathfrak{N}_p \mid i \in I).$$

Since $h_i(p)\mathfrak{N}_p \subseteq (\vee^\infty(h_i(p) \mid i \in I))\mathfrak{N}_p$, it follows that

$$l^\infty \operatorname{fit}\Big(\bigcup_{i\in I} h_i(p)\mathfrak{N}_p\Big) \subseteq l^\infty \operatorname{fit}(\vee^\infty(h_i(p) \mid i \in I)\mathfrak{N}_p) = (\vee^\infty(h_i(p) \mid i \in I))\mathfrak{N}_p$$

for any $p \in \mathbb{P}$. However, $\mathfrak{F} = \mathrm{LR}(t)$, where t is an l^∞-valued H-function such that

$$t(p) = (\vee^\infty(h_i(p) \mid i \in I))\mathfrak{N}_p$$

for any $p \in \mathbb{P}$. Hence, $f \le t$. Thus, $\mathfrak{M} \subseteq \mathfrak{F}$. Therefore, $\mathfrak{M} = \mathfrak{F}$. This proves the lemma. □

Lemma 6. *Let* $A \in \mathfrak{S} \cap \omega(\mathfrak{F}_1, \dots, \mathfrak{F}_m)$*, where* $\omega(x_1, \dots, x_m)$ *is a term of signature* $\{\cap, \vee^\infty\}$*, and let* $\mathfrak{F}_1, \dots, \mathfrak{F}_m$ *be some solvable totally local Fitting classes. Then there are groups* $A_1, \dots, A_m$ $(A_i \in \mathfrak{F}_i)$ *such that* $A \in \omega(l^\infty \operatorname{fit} A_1, \dots, l^\infty \operatorname{fit} A_m)$.

Proof. Let us prove by induction on the number r of occurrences of symbols belonging to $\{\cap, \vee^\infty\}$ in the term ω that there exist groups $A_i \in \mathfrak{F}_i$ $(i = 1, \dots, m)$ such that

$$A \in \omega(l^\infty \operatorname{fit} A_1, \dots, l^\infty \operatorname{fit} A_m).$$

For $r = 0$ we obviously have $A \in l^\infty \operatorname{fit} A$. Let us prove by induction on the nilpotent length of the group A that the assertion holds for $r = 1$.

Let $A \in \mathfrak{F}_1 \vee^\infty \mathfrak{F}_2 = l^\infty \operatorname{fit}(\mathfrak{F}_1 \cup \mathfrak{F}_2)$ and $\pi(A) = \{p_1, \dots, p_k\}$. For $l(A) = 1$ we have $A = P_1 \times \dots \times P_k$, where P_i is a Sylow p_i-subgroup of the group A. It is clear that $\pi(A) \subseteq \pi(\mathfrak{F}_1) \cup \pi(\mathfrak{F}_2)$. Let $p_1, \dots, p_j \in \pi(\mathfrak{F}_1)$ and $p_{j+1}, \dots, p_k \in \pi(\mathfrak{F}_2)$. Then $A_1 = P_1 \times \dots \times P_j \in \mathfrak{F}_1$ and $A_2 = P_{j+1} \times \dots \times P_k \in \mathfrak{F}_2$. Obviously,

$$A = A_1 \times A_2 \in (l^\infty \operatorname{fit} A_1) \vee^\infty (l^\infty \operatorname{fit} A_2).$$

We assume now that $l(A) > 1$. Let the desired assertion hold for all solvable groups whose nilpotent length is less than that of the group A. By Lemmas 1 and 2, for an arbitrary $p_i \in \pi(A)$ we have

$$F^{p_i}(A) \in f_1(p_i) \vee^\infty f_2(p_i) = (l^\infty \operatorname{fit}(F^{p_i}(G) \mid G \in \mathfrak{F}_1)) \vee^\infty (l^\infty \operatorname{fit}(F^{p_i}(G) \mid G \in \mathfrak{F}_2)),$$

where f_j is the minimal l^∞-valued H-function for the Fitting class $\mathfrak{F}_j$, $j = 1, 2$. Since $l(F^{p_i}(A)) < l(A)$, it follows by the induction assumption that there are groups $A_{i_1} \in f_1(p_i)$ and $A_{i_2} \in f_2(p_i)$ for which we have $F^{p_i}(A) \in (l^\infty \operatorname{fit} A_{i_1}) \vee^\infty (l^\infty \operatorname{fit} A_{i_2})$.

Let $B_{i_1} = A_{i_1} \wr Z_{p_i}$ and $B_{i_2} = A_{i_2} \wr Z_{p_i}$, where Z_{p_i} is a cyclic group of order p_i, and K is the base of the regular wreath product B_{i_1}. Since $A_{i_1} \in f_1(p_i)$, it follows by Lemma 4 that we have $B_{i_1} \in f_1(p_i)\mathfrak{N}_{p_i} \subseteq \mathfrak{F}_1$. Similarly, $B_{i_2} \in \mathfrak{F}_2$.

Hence, $A_1 = B_{1_1} \times \dots \times B_{t_1} \in \mathfrak{F}_1$ and $A_2 = B_{1_2} \times \dots \times B_{t_2} \in \mathfrak{F}_2$. Let us show that

$$A \in \mathfrak{F} = (l^\infty \operatorname{fit} A_1) \vee^\infty (l^\infty \operatorname{fit} A_2).$$

Let h be the minimal l^∞-valued H-function for the class $\mathfrak{F}$, and let f be an l^∞-valued H-function for the class $\mathfrak{F}$ such that $f(p) = h(p)\mathfrak{N}_p$ for any $p \in \mathbb{P}$. Let $i \in \{1, \dots, t\}$. Let us show that $F^{p_i}(A) \in f(p_i)$. First we prove that $A_{i_1}, A_{i_2} \in f(p_i)$. Assume that $A_{i_1} \notin f(p_i)$. In this case, since $f(p_i)$ is a Lockett class [13], it follows from [1, Chap. X, property 2.1 a)] that $(B_{i_1})_{f(p_i)} = K_1$, where K_1 is the base of the regular wreath product $(A_{i_1})_{f(p_i)} \wr Z_{p_i}$. By the properties of wreath products (see, e.g., [1, Chap. A, 18.2 d)]) we have

$$B_{i_1}/(B_{i_1})_{f(p_i)} = B_{i_1}/K_1 \simeq (A_{i_1}/(A_{i_1})_{f(p_i)}) \wr Z_{p_i}.$$

Hence, p_i is a divisor of the order of $B_{i_1}/(B_{i_1})_{f(p_i)}$.

On the other hand, since $B_{i_1} \in \mathfrak{F}$, it follows that

$$B_{i_1} \in \mathfrak{G}_{\pi(\mathfrak{F})} \cap \Big(\bigcap_{p \in \pi(\mathfrak{F})} f(p)\mathfrak{G}_{p'} \Big)$$

and, in particular, $B_{i_1} \in f(p_i)\mathfrak{G}_{p_i'}$. Hence, $B_{i_1}/(B_{i_1})_{f(p_i)} \in \mathfrak{G}_{p_i'}$. A contradiction. Thus, we have $A_{i_1} \in f(p_i)$. Similarly, $A_{i_2} \in f(p_i)$. Therefore, $l^\infty \operatorname{fit}(A_{i_1}, A_{i_2}) \subseteq f(p_i)$. Clearly $l^\infty \operatorname{fit}(A_{i_1}, A_{i_2}) = (l^\infty \operatorname{fit} A_{i_1}) \vee^\infty (l^\infty \operatorname{fit} A_{i_2})$. Thus, $F^{p_i}(A) \in f(p_i)$.

If $A \in \mathfrak{F}_1 \wedge^\infty \mathfrak{F}_2 = \mathfrak{F}_1 \cap \mathfrak{F}_2$, then $A \in (l^\infty \operatorname{fit} A) \wedge^\infty (l^\infty \operatorname{fit} A)$. This completes the proof of the theorem for $r = 1$.

Let a term ω have $r > 1$ occurrences of symbols belonging to $\{\cap, \vee^\infty\}$, and let the lemma hold for terms with lesser number of occurrences. Assume that ω is of the form

$$\omega_1(x_{i_1}, \dots, x_{i_a}) \Delta \omega_2(x_{j_1}, \dots, x_{j_b}),$$

where $\Delta \in \{\cap, \vee^\infty\}$ and $\{x_{i_1}, \dots, x_{i_a}\} \cup \{x_{j_1}, \dots, x_{j_b}\} = \{x_1, \dots, x_m\}$.

By $\mathfrak{H}_1$ we denote the Fitting class $\omega_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a})$ and by $\mathfrak{H}_2$ the Fitting class $\omega_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b})$. Then, as was proved above, there are groups $A_1 \in \mathfrak{H}_1$ and $A_2 \in \mathfrak{H}_2$ such that $A \in l^\infty \operatorname{fit} A_1 \Delta l^\infty \operatorname{fit} A_2$. Since the number of operations in the term ω_1 is less than r, it follows by the induction assumption that there are groups $B_1 \in \mathfrak{F}_{i_1}, \dots, B_a \in \mathfrak{F}_{i_a}$ such that $A_1 \in \omega_1(l^\infty \operatorname{fit} B_1, \dots, l^\infty \operatorname{fit} B_a)$. Similarly, there are groups $C_1 \in \mathfrak{F}_{j_1}, \dots, C_b \in \mathfrak{F}_{j_b}$ such that $A_2 \in \omega_2(l^\infty \operatorname{fit} C_1, \dots, l^\infty \operatorname{fit} C_b)$.

Let $x_{i_{t+1}}, \dots, x_{i_a} \in \{x_{j_1}, \dots, x_{j_b}\}$ and let $\{x_{i_1}, \dots, x_{i_t}\} \cap \{x_{j_1}, \dots, x_{j_b}\} = \varnothing$. Assume that

$$D_{i_k} = \begin{cases} B_k & \text{for } k < t+1, \\ B_k \times C_q, & \text{where } x_{i_k} = x_{j_q} \text{ for some } q \in \{1, \dots, b\} \text{ provided that } k \geq t+1. \end{cases}$$

Let $D_{j_k} = C_k$ if $x_{j_k} \notin \{x_{i_{t+1}}, \dots, x_{i_a}\}$. By $\mathfrak{M}_p$ we denote the class $l^\infty \operatorname{fit} D_{i_p}$, where $p = 1, \dots, a$, and by $\mathfrak{X}_c$ the class $l^\infty \operatorname{fit} D_{j_c}$, where $c = 1, \dots, b$.

Thus,

$$A_1 \in \omega_1(l^\infty \operatorname{fit} B_1, \dots, l^\infty \operatorname{fit} B_a) \subseteq \omega_1(l^\infty \operatorname{fit} D_{i_1}, \dots, l^\infty \operatorname{fit} D_{i_a}) = \omega_1(\mathfrak{M}_1, \dots, \mathfrak{M}_a),$$
$$A_2 \in \omega_2(l^\infty \operatorname{fit} C_1, \dots, l^\infty \operatorname{fit} C_b) \subseteq \omega_2(l^\infty \operatorname{fit} D_{j_1}, \dots, l^\infty \operatorname{fit} D_{j_b}) = \omega_2(\mathfrak{X}_1, \dots, \mathfrak{X}_b).$$

Thus, there exist Fitting classes $\mathfrak{R}_1, \dots, \mathfrak{R}_m$ such that

$$A \in \omega_1(\mathfrak{R}_{i_1}, \dots, \mathfrak{R}_{i_a}) \Delta \omega_2(\mathfrak{R}_{j_1}, \dots, \mathfrak{R}_{j_b}) = \omega(\mathfrak{R}_1, \dots, \mathfrak{R}_m),$$

where $\mathfrak{R}_i = l^\infty \operatorname{fit} K_i$ for $K_i \in \mathfrak{F}_i$. This proves the lemma. □

Lemma 7. *Let $\omega(x_1, \dots, x_m)$ be a term of signature $\{\cap, \vee^\infty\}$, and let f_i be an inner l^∞-valued H-function for a Fitting class $\mathfrak{F}_i$, $i = 1, \dots, m$. Then*

$$\omega(\mathfrak{F}_1, \dots, \mathfrak{F}_m) = \mathrm{LR}(\omega(f_1, \dots, f_m)).$$

Proof. Let us perform induction on the number r of occurrences of the symbols belonging to $\{\cap, \vee^\infty\}$ in the term ω. Let

$$\omega(x_1, \dots, x_m) = \omega_1(x_{i_1}, \dots, x_{i_a}) \Delta \omega_2(x_{j_1}, \dots, x_{j_b}),$$

where $\Delta \in \{\cap, \vee^\infty\}$ and $\{x_{i_1}, \dots, x_{i_a}\} \cup \{x_{j_1}, \dots, x_{j_b}\} = \{x_1, \dots, x_m\}$. Assume that the lemma holds for the terms ω_1 and ω_2. Then

$$\omega_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}) = \mathrm{LR}(\omega_1(f_{i_1}, \dots, f_{i_a})), \qquad \omega_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b}) = \mathrm{LR}(\omega_2(f_{j_1}, \dots, f_{j_b})).$$

It is clear that $\omega_1(f_{i_1}, \dots, f_{i_a})$ and $\omega_2(f_{j_1}, \dots, f_{j_b})$ are inner l^∞-valued H-functions for the Fitting classes $\omega_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a})$ and $\omega_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b})$, respectively. Hence, by induction we obtain

$$\begin{aligned} \omega(\mathfrak{F}_1, \dots, \mathfrak{F}_m) &= \omega_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}) \Delta \omega_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b}) \\ &= \mathrm{LR}(\omega_1(f_{i_1}, \dots, f_{i_a}) \Delta \omega_2(f_{j_1}, \dots, f_{j_b})) = \mathrm{LR}(\omega(f_1, \dots, f_m)), \end{aligned}$$

where $\Delta \in \{\cap, \vee^\infty\}$. This proves the lemma. □

An element c of a complete lattice L is said to be *compact* if, for any subset $X \subseteq L$, it follows from the inequality $c \leq \sup_L X$ that there exists a finite subset $X_0 \subseteq X$ such that $c \leq \sup_L X_0$.

Lemma 8. *Let $\mathfrak{F} = l^\infty \operatorname{fit} G$, where the group G is solvable. Then $\mathfrak{F}$ is a compact element in the lattice l^∞.*

Proof. Let us show by induction on the nilpotent length of the group G that $\mathfrak{F}$ is a compact element of l^∞. Let $\mathfrak{F} \subseteq \mathfrak{M} = \vee^\infty(\mathfrak{F}_i \mid i \in I)$, where $\mathfrak{F}_i$ is a totally local Fitting class.

If $l(G) = 1$, then $G = P_1 \times \cdots \times P_k$, where P_i is a Sylow p_i-subgroup of the group G. Hence, there are indices $j_1, \ldots, j_k \in I$ such that $p_i \in \pi(\mathfrak{F}_{j_i})$, i.e., $P_i \in \mathfrak{F}_{j_i}$. Therefore, $G \in \mathfrak{F}_{j_1} \vee^\infty \cdots \vee^\infty \mathfrak{F}_{j_k}$. Hence, $\mathfrak{F} \subseteq \mathfrak{F}_{j_1} \vee^\infty \cdots \vee^\infty \mathfrak{F}_{j_k}$.

Let $l(G) > 1$, and let all totally local Fitting classes of the form $l^\infty \operatorname{fit} A$, where A is a solvable group and $l(A) < l(G)$, are compact elements of the lattice l^∞. Let f_i be the minimal l^∞-valued H-function for the class $\mathfrak{F}_i$, let f be the minimal l^∞-valued H-function for the class $\mathfrak{F}$, and let m be the minimal l^∞-valued H-function for the class $\mathfrak{M}$. Then by Lemma 1 we have $f(p) = l^\infty \operatorname{fit}(F^p(G))$ for all $p \in \pi(G)$, and $f(p) = \varnothing$ for $p \in \mathbb{P} \setminus \pi(G)$. Moreover, it follows from Lemma 1 that $f \leq m$. By Lemma 2, $m = \vee^\infty(f_i \mid i \in I)$. Since $l(F^p(G)) < l(G)$, it follows from the induction assumption that, for any $p \in \pi(G)$, there are indices $i_1, \ldots, i_t \in I$ for which

$$F^p(G) \in f_{i_1}(p) \vee^\infty \cdots \vee^\infty f_{i_t}(p).$$

Since $|\pi(G)| < \infty$, it follows from the last relation that there are indices $j_1, \ldots, j_k \in I$ such that $G \in \mathfrak{F}_{j_1} \vee^\infty \cdots \vee^\infty \mathfrak{F}_{j_k}$. Thus, $\mathfrak{F} \subseteq \mathfrak{F}_{j_1} \vee^\infty \cdots \vee^\infty \mathfrak{F}_{j_k}$. Therefore, $\mathfrak{F}$ is a compact element of the lattice l^∞. This proves the lemma. □

By $L^\infty(\mathfrak{F})$ we denote (see [6]) the lattice of all totally local Fitting subclasses in $\mathfrak{F}$.

Lemma 9. *Let $\mathfrak{F}$ be a totally local Fitting class. Then, for any positive integer $k \geq 2$, the lattices $L^\infty(\mathfrak{F}\mathfrak{N}^{k-1})$ and $L^\infty(\mathfrak{F}\mathfrak{N}^k)$ generate the same variety of lattices.*

Proof. Let us choose an identity

$$\omega_1(x_{i_1}, \ldots, x_{i_a}) = \omega(x_{j_1}, \ldots, x_{j_b}) \tag{$*$}$$

of signature $\{\cap, \vee^\infty\}$.

If identity $(*)$ holds in the lattice $L^\infty(\mathfrak{F}\mathfrak{N}^k)$, then it holds in any sublattice of the lattice $L^\infty(\mathfrak{F}\mathfrak{N}^k)$. Therefore, identity $(*)$ holds in the lattice $L^\infty(\mathfrak{F}\mathfrak{N}^{k-1})$.

We assume now that identity $(*)$ holds in the lattice $L^\infty(\mathfrak{F}\mathfrak{N}^{k-1})$ and that $\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}$ and $\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b}$ are some Fitting classes in $L^\infty(\mathfrak{F}\mathfrak{N}^k)$. Let f_{i_c} be the minimal l^∞-valued H-function for the class $\mathfrak{F}_{i_c}$, $c = 1, \ldots, a$, and let f_{j_d} be the minimal l^∞-valued H-function for the class $\mathfrak{F}_{j_d}$, $d = 1, \ldots, b$. By Lemma 7,

$$\omega_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}) = \mathrm{LR}(\omega_1(f_{i_1}, \ldots, f_{i_a})), \qquad \omega_2(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b}) = \mathrm{LR}(\omega_2(f_{j_1}, \ldots, f_{j_b})).$$

Note that, for any $p \in \mathbb{P}$, the classes

$$f_{i_1}(p), \ldots, f_{i_a}(p), \qquad f_{j_1}(p), \ldots, f_{j_b}(p)$$

belong to the lattice $L^\infty(\mathfrak{F}\mathfrak{N}^{k-1})$. Hence,

$$\omega_1(f_{i_1}, \ldots, f_{i_a})(p) = \omega_1(f_{i_1}(p), \ldots, f_{i_a}(p)) = \omega_2(f_{j_1}(p), \ldots, f_{j_b}(p)) = \omega_2(f_{j_1}, \ldots, f_{j_b})(p).$$

Therefore, $\omega_1(\mathfrak{F}_{i_1}, \ldots, \mathfrak{F}_{i_a}) = \omega_2(\mathfrak{F}_{j_1}, \ldots, \mathfrak{F}_{j_b})$. Thus, identity $(*)$ holds in the lattice $L^\infty(\mathfrak{F}\mathfrak{N}^k)$. This proves the lemma. □

Lemma 10. *Let η be a sublattice of the lattice of solvable totally local Fitting classes that contains all singly generated totally local Fitting subclasses of any Fitting class $\mathfrak{F} \in \eta$. Then the identity $\omega_1 = \omega_2$ of signature $\{\cap, \vee^\infty\}$ holds in η provided that it holds in any singly generated totally local Fitting classes in η.*

Proof. Let $x_{i_1}, \dots, x_{i_a}$ be the variables appearing in the term ω_1, let $x_{j_1}, \dots, x_{j_b}$ be the variables appearing in the term ω_2, and let $\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}, \mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b} \in \eta$. Let us show that

$$\mathfrak{F} = \omega_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a}) \subseteq \omega_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b}) = \mathfrak{M}.$$

Let $x_{j_1}, \dots, x_{j_t} \in \{x_{i_1}, \dots, x_{i_a}\}$, and let $\{x_{j_{t+1}}, \dots, x_{j_b}\} \cap \{x_{i_1}, \dots, x_{i_a}\} = \varnothing$. Assume that $A \in \mathfrak{F}$. In this case, by Lemma 6, there are groups $A_{i_1}, \dots, A_{i_a}$ such that $A_{i_k} \in \mathfrak{F}_{i_k}$ for all $k \in \{1, \dots, a\}$ and

$$A \in \mathfrak{S} \cap \omega_1(l^\infty \operatorname{fit} A_{i_1}, \dots, l^\infty \operatorname{fit} A_{i_a}).$$

Let $\mathfrak{H}_{i_k} = l^\infty \operatorname{fit} A_{i_k}$, and let

$$\mathfrak{H}_{j_k} = \begin{cases} \mathfrak{H}_{i_c}, & \text{where } x_{j_k} = x_{i_c} \text{ for some } c \in \{1, \dots, a\} \text{ for all } k \in \{1, \dots, t\}, \\ l^\infty \operatorname{fit} B_{j_k} & \text{for some group } B_{j_k} \in \mathfrak{F}_{j_k} \text{ provided that } k > t. \end{cases}$$

By assumption, $\omega_1(\mathfrak{H}_{i_1}, \dots, \mathfrak{H}_{i_a}) = \omega_2(\mathfrak{H}_{j_1}, \dots, \mathfrak{H}_{j_b}) \subseteq \mathfrak{M}$. Therefore, $A \in \mathfrak{M}$. Thus, $\mathfrak{F} \subseteq \mathfrak{M}$. The converse inclusion can be proved in a similar way. This proves the lemma. □

Recall [6] that, if classes of groups $\mathfrak{M}$ and $\mathfrak{H}$ are such that $\mathfrak{M} \cap \mathfrak{H} = (1)$, then $\mathfrak{M} \oplus \mathfrak{H}$ stands for the set of all groups of the form $\{A \times B \mid A \in \mathfrak{M}, B \in \mathfrak{H}\}$.

Theorem. *The lattice of all solvable totally local Fitting classes is algebraic and distributive and each its element that differs from* (1) *and* $\mathfrak{S}$ *is not complementable in this lattice.*

Proof. Let us show that the lattice $L^\infty(\mathfrak{S})$ is algebraic. Obviously, any totally local Fitting class is the union of its singly generated totally local Fitting subclasses in the lattice l^∞. Let $\mathfrak{F} = l^\infty \operatorname{fit} G$, where the group G is solvable. By Lemma 8, $\mathfrak{F}$ is a compact element of the lattice l^∞. Hence, $\mathfrak{F}$ is a compact element of the sublattice $L^\infty(\mathfrak{S})$ of the lattice l^∞ as well. Thus, the lattice of all solvable totally local Fitting classes is algebraic, and its compact elements are the singly generated totally local Fitting classes.

Let us prove now that the lattice $L^\infty(\mathfrak{S})$ is distributive. We first show by induction on r that the lattice $L^\infty(\mathfrak{N}^r)$ is distributive. The lattice $L^\infty(\mathfrak{N})$ is certainly distributive. Let $r > 1$, and let the lattice $L^\infty(\mathfrak{N}^{r-1})$ be distributive. Then it follows from Lemma 9 that the lattice $L^\infty(\mathfrak{N}^r)$ is also distributive.

Assume now that $\mathfrak{F} = l^\infty \operatorname{fit} G$, where $l(G) = r$. Then $G \in \mathfrak{N}^r$, and therefore $L^\infty(l^\infty \operatorname{fit} G) \subseteq L^\infty(\mathfrak{N}^r)$. Thus, the lattice $L^\infty(l^\infty \operatorname{fit} G)$ is distributive. Hence, by Lemma 10, the lattice of all solvable totally local Fitting classes is distributive.

Let us prove now that each solvable totally local Fitting class that differs from (1) and $\mathfrak{S}$ is not complementable in the lattice $L^\infty(\mathfrak{S})$. Let $\mathfrak{M}$, where $\mathfrak{M} \neq (1)$ and $\mathfrak{M} \neq \mathfrak{S}$, be a solvable totally local Fitting class, and let $\mathfrak{H}$ be a complement to $\mathfrak{M}$ in the lattice $L^\infty(\mathfrak{S})$. Then $\mathfrak{S} = \mathfrak{M} \vee^\infty \mathfrak{H}$ and $\mathfrak{M} \cap \mathfrak{H} = (1)$.

Let us show that $\mathfrak{M} \vee^\infty \mathfrak{H} = \mathfrak{M} \vee \mathfrak{H}$. Consider the Fitting class $\mathfrak{F} = \mathfrak{M} \vee \mathfrak{H}$. Since $\mathfrak{M} \cap \mathfrak{H} = (1)$, it follows that $\mathcal{K}(\mathfrak{M}) \cap \mathcal{K}(\mathfrak{H}) = \varnothing$, where $\mathcal{K}(\mathfrak{M})$ is the set of all composition factors of the groups in $\mathfrak{M}$ and $\mathcal{K}(\mathfrak{H})$ is the set of all composition factors of the groups in $\mathfrak{H}$. Hence, $\mathfrak{F} = \mathfrak{M} \oplus \mathfrak{H}$ by Lemma 4 from [14]. Let us show that the class $\mathfrak{M} \oplus \mathfrak{H}$ is totally local. Let m and h be the minimal l^∞-valued H-functions of the classes $\mathfrak{M}$ and $\mathfrak{H}$, respectively. Let f be an H-function such that

$$f(p) = \begin{cases} m(p) & \text{for } p \in \pi(\mathfrak{M}), \\ h(p) & \text{for } p \in \pi(\mathfrak{H}), \\ \varnothing & \text{for } p \in \mathbb{P} \setminus (\pi(\mathfrak{M}) \cup \pi(\mathfrak{H})). \end{cases}$$

Let us show that $\mathfrak{F} = \mathrm{LR}(f)$. Let G be a group of minimal order in $\mathrm{LR}(f) \setminus \mathfrak{F}$. Then the group G is comonolithic, and its comonolith is $M = G_{\mathfrak{F}}$. Since $G \in \mathrm{LR}(f)$, it follows that $F^p(G) \in f(p)$ for all $p \in \pi(G)$. Hence, if $p \in \pi(G)$, then it follows from the construction of the H-function f that either $f(p) = m(p) \neq \varnothing$ or $f(p) = h(p) \neq \varnothing$. Thus, $\pi(G) \subseteq \pi(\mathfrak{M}) \cup \pi(\mathfrak{H})$.

Let $p \in \pi(G/M)$. Then $p \in \pi(\mathfrak{M}) \cup \pi(\mathfrak{H})$. Assume now that $p \in \pi(\mathfrak{M})$. Hence, G/M is a p-group, and $F^p(G) = O^p(G) \in f(p) = m(p)$. Hence, $G \in \mathfrak{M} \subseteq \mathfrak{F}$ by Lemma 4. A contradiction. Thus, $\mathrm{LR}(f) \subseteq \mathfrak{F}$.

Assume that the converse inclusion fails and that G is a group of minimal order in $\mathfrak{F} \setminus \mathrm{LR}(f)$. Then the group G is comonolithic. Therefore, either $G \in \mathfrak{M}$ or $G \in \mathfrak{H}$. Let $G \in \mathfrak{M} = \mathrm{LR}(m)$. Thus, $F^p(G) \in m(p) = f(p)$ for any $p \in \pi(G)$. Hence, $G \in \mathrm{LR}(f)$. This means that $\mathfrak{F} \subseteq \mathrm{LR}(f)$. Therefore, $\mathfrak{F} = \mathrm{LR}(f)$ is a totally local Fitting class. However, $\mathfrak{M} \vee \mathfrak{H} \subseteq \mathfrak{F}$. This means that $\mathfrak{M} \vee^\infty \mathfrak{H} = \mathfrak{M} \oplus \mathfrak{H} = \mathfrak{M} \vee \mathfrak{H}$. Hence, for any comonolithic group $G \in \mathfrak{S}$ we have either $G \in \mathfrak{M}$ or $G \in \mathfrak{H}$.

Let Z_p and Z_q be some groups of orders p and q, respectively, where $p \in \pi(\mathfrak{M})$ and $q \in \pi(\mathfrak{H})$. Let $r \in \mathbb{P} \setminus \{p, q\}$. Then, by [1, Chap. B, Corollary 10.7], the group $A = Z_p \times Z_q$ has a simple faithful module P over a field $\mathbb{F}_r$ with r elements. Let $B = [P]A$. Then $B \in \mathfrak{S}$ and at the same time $B \notin \mathfrak{M}$ because $q \notin \pi(\mathfrak{M})$, and $B \notin \mathfrak{H}$ because $p \notin \pi(\mathfrak{H})$. However, the group B is comonolithic, and hence either $B \in \mathfrak{M}$ or $B \in \mathfrak{H}$. A contradiction. Thus, any nonzero and nonidentity element of the lattice of all solvable totally local Fitting classes is not complementable in this lattice. This proves the theorem. □

Let L be an arbitrary lattice. By a *pseudocomplement of an element a with respect to an element b* we mean [15] the largest of the elements x of the lattice L that satisfy the inequality $a \wedge x \leq b$. The pseudocomplement of an element a with respect to an element b is denoted by $a * b$. By the *pseudocomplement of an element a in a lattice with zero* we mean the relative pseudocomplement $a * 0$. By definition, a *lattice with pseudocomplements* is a lattice with zero in which each element has the pseudocomplement.

By Corollary 2 to Theorem 1 in [16, Chap. II, p. 151 of the Russian translation], the above theorem has the following consequence.

Corollary 1. *The lattice of all solvable totally local Fitting classes is a lattice with pseudocomplements.*

Recall that a representation of an element a in the form $x_0 \vee \cdots \vee x_{n-1}$ is said to be *cancelable* [16] if $a = x_0 \vee \cdots \vee x_{i-1} \vee x_{i+1} \vee \cdots \vee x_{n-1}$ for some $0 \leq i < n$; otherwise it is said to be *noncancelable.*

A totally local Fitting class $\mathfrak{F}$ is said to be *l^∞-irreducible* [2] if the class $\mathfrak{F}$ cannot be represented in the form $\mathfrak{F} = \vee^\infty(\mathfrak{F}_i \mid i \in I)$, where $\{\mathfrak{F}_i \mid i \in I\}$ is the set of all proper totally local Fitting subclasses in $\mathfrak{F}$.

Corollary 2. *Let $\mathfrak{F}$ be a solvable singly generated totally local Fitting class. Then $\mathfrak{F}$ has a unique representation in the form of an noncancelable union $\mathfrak{F}_1 \vee^\infty \cdots \vee^\infty \mathfrak{F}_t$ of some its totally local l^∞-irreducible Fitting subclasses $\mathfrak{F}_1, \dots, \mathfrak{F}_t$.*

Proof. By Corollary 13 to Theorem 9 in [16, Chap. II], to prove this corollary, it suffices to show that the lattice $L^\infty(\mathfrak{F})$ of all totally local Fitting subclasses of an arbitrary solvable singly generated totally local Fitting class $\mathfrak{F} = l^\infty \operatorname{fit} G$ is finite. Let us perform induction on the nilpotent length of the group G.

If $l(G) = 1$, then any nonidentity totally local Fitting subclass in $\mathfrak{F}$ is of the form $\mathfrak{N}_\pi$, where $\pi \subseteq \pi(\mathfrak{F})$. Hence, there are only finitely many totally local Fitting subclasses of $\mathfrak{F}$.

Let $l(G) > 1$, and let the lattice $L^\infty(l^\infty \operatorname{fit} A)$ be finite for any totally local Fitting class of the form $l^\infty \operatorname{fit} A$, where $l(A) < l(G)$. Let $\mathfrak{M}$ be an arbitrary totally local Fitting subclass in $\mathfrak{F}$, and let m and f be the minimal l^∞-valued H-functions of the classes $\mathfrak{M}$ and $\mathfrak{F}$, respectively. Then it follows from Lemma 1 that $m \leq f$. Moreover, by the same lemma, for any $p \in \pi(\mathfrak{F})$ we have the relation

$$f(p) = l^\infty \operatorname{fit}(F^p(A) \mid A \in \mathfrak{F}).$$

Since $l(F^p(G)) < l(G)$, it follows that the lattice $L^\infty(f(p))$ is finite by the induction assumption. Since the set $\pi(\mathfrak{F})$ is also finite, it follows that $\mathfrak{F}$ has only finitely many totally local Fitting subclasses. This proves the corollary. □

For any two totally local Fitting classes $\mathfrak{M}$ and $\mathfrak{H}$, where $\mathfrak{M} \subseteq \mathfrak{H}$, by $\mathfrak{H}/^\infty\mathfrak{M}$ we denote [6] the lattice of totally local Fitting classes between $\mathfrak{M}$ and $\mathfrak{H}$.

Since any distributive lattice is modular, the above theorem implies the following assertion.

Corollary 3. *For any two solvable totally local Fitting classes $\mathfrak{M}$ and $\mathfrak{H}$, the following lattice isomorphism exists:*

$$\mathfrak{M} \vee^\infty \mathfrak{H}/^\infty\mathfrak{M} \simeq \mathfrak{H}/^\infty\mathfrak{H} \cap \mathfrak{M}.$$

Corollary 4 [6, Theorem 4.1.7]. *The lattice of all solvable totally local formations is distributive.*

Proof. Let us show that any solvable totally local Fitting class $\mathfrak{F}$ is hereditary. Without loss of generality, we may assume that $\mathfrak{F} = l^\infty \operatorname{fit} G$ for some solvable group G. If the group G is nilpotent, then the assertion is obvious. Assume that the nilpotent length t of G exceeds one. Then, if f is the minimal l^∞-valued H-function for the class $\mathfrak{F}$, then, since $l(F^p(G)) < t$, it follows from the induction assumption that, for any $p \in \pi(G)$, the Fitting class $f(p) = l^\infty \operatorname{fit}(F^p(G))$ is hereditary. Hence, the class $\mathfrak{F}$ is hereditary, and therefore, by the results in [17], the class $\mathfrak{F}$ is a totally local formation. Therefore, each solvable totally local Fitting class is a totally local formation.

Arguing in the same way, we can readily see that each solvable totally local formation is a hereditary Fitting class. However, by the theorem in [7], any solvable hereditary Fitting class is a totally local Fitting class. Hence, any solvable totally local formation is a totally local Fitting class.

Thus, the lattices $L^\infty(\mathfrak{S})$ and $L_\infty(\mathfrak{S})$ coincide. Therefore, by the above theorem, the lattice of all solvable totally local formations is distributive. This proves the corollary. □

The technique of proofs and the arguments suggested in the present paper heavily depend on the solvability condition. As far as the nonsolvable case is concerned, we can say nothing even on the modularity of the lattice of all totally local Fitting classes.

Question. Is the lattice of all totally local Fitting classes distributive (or at least modular)?

A similar question for totally local formations was presented in [6, Question 4.2.14].

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